A STUDY MANUAL

ON

SELECTED TOPICS IN

OPERATIONS RESEARCH

" Quantitative Approaches to Decision Making"

Ilembo, Bahati

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Bahati Ilembo

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MZUMBE UNIVERSITY

DEDICATION

This Study Manual is dedicated to my beloved biological mother Theresia Nimes Maganga who has natural knowledge in Mathematics.

TABLE OF CONTENTS

Dedication i
Preface v
Preface....v
CHAPTER ONE....1
MANAGEMENT SCIENCE APPROACH TO PROBLEM SOLVING....1
1.0 Background....1
1.1 Phases of Operations Research/Management Science....2
CHAPTER TWO....7
LINEAR PROGRAMMING....7
2.0 Introduction....7
2.1 Definition....7
2.2 FORMULATION OF LINEAR PROGRAMMING MODELS....8
2.3 SOLUTION METHODS TO LINEAR PROGRAMMING PROBLEMS....15
2.3.1 GRAPHICAL APPROACH....15
2.3.2 SIMPLEX METHOD....18
2.4 TIES IN THE SELECTION OF THE NON BASIC VARIABLE....24
2.5 TIES IN THE MINIMUM RATIO RULE AND DEGENERACY....24
2.6 THE BIG M Method....24
2.7 THE TWO PHASE SIMPLEX METHOD....28
2.7.1 Phase 1....28
2.7.2 Phase II....28
2.8 SENSITIVITY ANALYSIS IN LINEAR PROGRAMMING.31
2.9 DUALITY THEOREM....37
2.9.1 Formulation of a Dual problem....39
2.10 ECONOMIC INTERPRETATION OF THE DUAL VARIABLES....45
2.11 RELAȚIONSHIP BETWEEN DUAL AND PRIMAL TABLEAU....48
Exercise....49

CHAPTER THREE..52
TRANSPORTATION AND ASSIGNMENT PROBLEMS..........52
3.0 Introduction..52
3.1 Statement of the problem..52
3.2 Definition...53
3.3 VARIATIONS OF TRANSPORTATION PROBLEM..........54
3.4 SOLUTION METHODS TO TRANSPORTATION PROBLEMS..55
3.4.1 MINIMUM COST METHOD.....................................56
3.4.2 NORTHWEST CORNER METHOD..........................58
3.4.3 THE VOGEL APPROXIMATION METHOD-VAM.........61
3.5 IMPROVING BASIC FEASIBLE SOLUTION TO OPTIMALITY..67
3.5.1 Loop Determination..70
3.6 DEGENERATE BASIC FEASIBLE SOLUTION.............73
3.7 ASSIGNMENT PROBLEM...75
3.7.2 Representation...76
3.7.3 Unbalanced Assignment Problem..........................77
3.7.4 SOLUTION TECHNIQUES......................................78
3.7.1 Introduction 75
3.7.4.1 KUHN'S Method (1950)...78
3.8 EFFECTIVE MATRIX WITH NEGATIVE ELEMENTS.....88
3.8.1 ROW/COLUMN APPROACH.....................................88
3.8.2 MATRIX APPROACH..88
Exercise..89
CHAPTER FOUR...92
INVENTORY CONTROL...92
4.0 Introduction...92
4.1 Types of stock held by business and industry and purpose for which they are held...................................93
4.1.1 Raw material stock...93
4.1.2 Work- in- progress or stock-in-process...................93
4.1.3 Finished goods..94
4.2 THE CLASSICAL ECONOMIC ORDER QUANTITY (EOQ)..96
4.3 PRODUCTION LOT SIZE MODEL..............................103

4.4 ECONOMIC ORDER QUANTITY MODEL WITH SHORTAGES...107
Exercise..116
CHAPTER FIVE...118
QUEUING THEORY...118
5.0 Introduction...118
5.1 Description of a Queueing System..........................119
5.2 PARAMETERS IN QUEUING SYSTEMS AND NOTATIONS...122
5.3 Kendal's Notation...122
5.4 SINGLE SERVER SYSTEM...123
5.5 MEASURES OF PERFOMANCE...................................126
5.6 FINITE BUFFER SIZE M/M/N....................................133
5.7 INTRODUCTION TO MULTI-SERVER QUEUEING SYSTEM..136
5.8 M/G/1 Model...142
5.9 VARIATIONS OF M/G/1 MODEL...............................142
5.10 JACKSON'S NETWORK...143
CHAPTER SIX..149
NETWORK ANALYSIS..149
6.0 Introduction...149
6.1 CPM/ PERT Network Components.............................150
6.2 Classification of activities..151
6.3 CRITICAL PATH METHOD (CPM)...............................151
Definition..152
6.3.1 Methodology of CPM..152
6.4 Critical Path Analysis..152
6.4.1 Time Estimates in Network Analysis........................153
6.6 PROGRAMME EVALUATION AND REVIEW TECHNIQUE (PERT)...156
6.6.1 Advantages and disadvantages of PERT.................157
6.7 Worked Examples..159
ANSWERS TO EXERCISES...178
Reference..191

Preface

Before writing this manuscript, I tried to recall some of the reasons that made me dislike some of the operations research books that I had used when I was an undergraduate student. One prominent reason was that those texts consisted of long chapters that tried to explain everything about quantitative techniques without giving many examples. Even when the examples are given, one had to make a lot of efforts in order to follow those examples because you might find a lot of steps have been skipped or jumped. This one is centered on simples and straightforward examples (solved step by step) that demonstrate in detail the fundamentals of the techniques. It is my sincere hope that for all my target readers, specifically those students who, among other courses, pursue courses in Operations Research or Management Science (as applied by other scholars), can easily apply the same solution steps to their homework and test problems. One may join hand with me in the fact that the manuscript will serve the best for all students with limited mathematical backgrounds. As you read, you will find that topics that you thought would be very hard are presented in such a way that they are not that difficult.

The text encompasses five topics, namely, Management Science Approaches to Problems Solving, Linear Programming, Transportation and Assignment Problems, Inventory Control, Queuing Theory and Network Analysis. The topics are presented in such a way that readers will acquaint themselves with suitable tools to analyze problems that are directly related to Operations research using the contained quantitative approaches. The simple and complex examples will serve as guidelines to students when they face challenges from other sources. In general, this text is expected not only to teach you specific techniques but also provide a method for approaching problems that will be very useful in your endeavor. It is indeed user-friendly.

CHAPTER ONE

MANAGEMENT SCIENCE APPROACH TO PROBLEM SOLVING

1.0 Background

Historical perspective on the development of the field of operations research (Management science) can be looked at the following instance.

Although a number of the mathematical techniques that make up management science/operations research date back to the turn of the century or before, the field of operations research itself can trace its beginnings to military operations research groups formed during World War II in Great Britain 1939.The groups typically consisted of a team of about a dozen individuals from different fields of science, mathematics and the military brought together to find solutions to military-related problems. One of the most famous of these groups – called "Blackett's circus" after its leader, Nobel Laureate P.M.S. Blackett of the University of Manchester and a former naval officer- included three physiologist, two mathematical physicists, one astrophysicists, one general physicist, two mathematicians, an army officer and a surveyor.

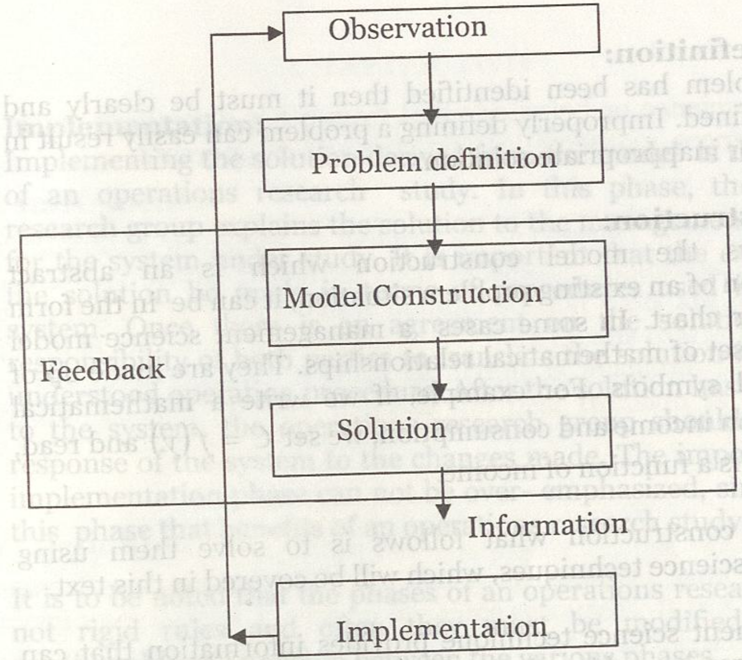
Blackette's group and other Operation Research teams made significant contributions in improving Britain's early warning radar system(which was instrumental in their victory in the battle of Britain) , aircraft gunnery, antisubmarine warfare, civilian defense, convoy size determination, and bombing raids over German. The successes achieved by the British Operations research groups were observed by two Americans working for the United States military,Dr.James B.Conant and Dr.Vannevar Bush, who recommended that OR teams be established in the US branches of the military. Subsequently ,both the Air force and Navy created operations research groups.

After World War II, the contributions of the operations research groups were considered to be so valuable that the Army, Air Force and Navy set up various agencies to continue research of military problems. Two of the more famous agencies were the Navy's Operations Evaluation group at M.I.T and project RAND established by the Air Force to study aerial warfare. Many of the individuals who developed operations research and/or Management science techniques did so while working at one of these agencies after World War II or as a result of their work there.

As the war ended and the mathematical models and techniques that were kept secret during the war, began to be released, there was a natural inclination to test their applicability to business problems. At the same time, various consulting firms were established to apply these techniques to industrial and business problems, and courses in the use of quantitative techniques for business management began to surface in American Universities. In the Early 1950's the use of these quantitative techniques to solve management problems began to be known as management science, and it was popularized by a book of that name by Stafford Beer of Great Britain.

1.1 Phases of Operations Research/Management Science

Management science encompasses a special, systematic approach to problem solving, which closely parallels what is known as the scientific method for attacking problems. The approach follows a general recognized, ordered set of steps which can be summarised in tabular form and we will analyze each step individually:



Let us discuss on the above phases of operations research / management science:

Observation:

The process involves identification of a problem that exists in the system (organization).The system must be continuously and closely observed so that problems can be identified as soon as they occur or are anticipated. Problems are not always the result of a crisis that must be reacted to, but instead, frequently involve an anticipatory or planning situation.

Problem definition:
Once a problem has been identified then it must be clearly and concisely defined. Improperly defining a problem can easily result in no solution or inappropriate solution.

Model construction:
Then follows the model construction which is an abstract representation of an existing problem situation, it can be in the form of a graph or chart. In some cases a management science model consists of a set of mathematical relationships. They are made up of numbers and symbols. For example, if we write a mathematical relationship on income and consumption, we set $C = f(y)$ and read, consumption is a function of income.

After model construction what follows is to solve them using management science techniques, which will be covered in this text.

The management science technique provides information that can aid the manager in making decision. But the decision maker must contemplate the results before implementing them. Decision maker must combine the information obtained with his or her own expertise and experience.

Deriving the solution:
Once the data has been collected and inputs prepared, the next step is to obtain a solution to the problem from the model. This is accomplished by determining the optimum solution to the model and then applying the solution to the actual problem. Sometimes mathematical complexities in the model make an optimum solution impossible and a "good" answer must suffice. Since a model is an approximation of a real system or problem, even the optimum solution to the model does not guarantee an optimum solution for the real problem. However, if the model has been well formulated and tested, solutions from the model will provide a good approximation to the optimum solution.

Implementation:
Implementing the solution derived from the model is the last phase of an operations research study. In this phase, the operations research group explains the solution to the management responsible for the system under study. It is important that the explanation of the solution be made in terms of procedures used in the actual system. Once there is an agreement on the solution, it is the responsibility of both parties to translate the solution into an easily understood operating procedure. After the solution has been applied to the system, the operations research group should observe the response of the system to the changes made. The importance of the implementation phase can not be over- emphasized, since it is from this phase that benefits of an operations research study are realized.

It is to be noted that the phases of an operations research study are not rigid rules and often they must be modified, there is a considerable relationship between the various phases.
Management science/operations research is an ongoing process. Completion of the five steps described above does not necessarily mean that the management science process has been completed. The model results and the decision based on the results provide feedback to the original model. The original management science model can then be modified to test different condition and decision the manager thinks might occur in the future, or the results may indicate that a problem exists that had not been considered previously; if so the original model can be altered or reconstructed. Because models can be modified or reconstructed, the management science process can be continuous rather that simply providing one solution to one problem.

Definition:
Management Science or Operations research is the application of a scientific approach to solving management problems in order to help managers make better decisions (Example of management problem can be making use of available resources given some constraints and of the like). It encompasses a number of mathematically oriented techniques that have been either developed within the field of

management science or being adopted from other disciplines, such as natural sciences, mathematics, statistics and engineering.

CHAPTER TWO

LINEAR PROGRAMMING

2.0 Introduction

Programming problems in general are concerned with the use or allocation of scarce resources such as labour, materials, machines and capital in the best possible manner so that costs are minimized or profit is maximized. The term "best" here means that some choice or a set of alternative courses of action is available for making decision. Looking at this context, the best decision is found by solving a mathematical problem.

2.1 Definition

Linear Programming (LP) is the mathematical procedure for determining optimal allocation of scarce resources. It is a procedure which has found practical applications in almost all faces of business from advertising to production planning. Transportation, distribution and aggregate production planning problems are the most typical objects of linear programming analysis.

It deals with linear relationships or class of programming problems, which both the objective function to be optimized is linear and all the relations among the variables corresponding to resources are linear (by saying linear, we mean all the variables included in the equations or constraints are raised to power one, otherwise it is non-linear).

Any Linear programming problem consists of the objective function and a set of constraints. In most cases constraints come from the environment in which you work to achieve your objectives or one's objective. So, we have in any linear programming problems the objective function (called the measure of performance) which must be linear and so do the constraints.

Linear programming techniques are widely used to solve a number of military, economic, industrial and social problems. Three primary reasons for its wide use are;

1. A large variety of problems in diverse fields can be represented or at least approximated as linear programming models.
2. Ease through which data variation (sensitivity analysis) can be handled through Linear Programming models
3. Efficient techniques for solving linear programming problems are available.

At this point, we should point out that the solution procedure are iterative in nature, and hence even for the moderate size problems one has to resort to a digital computer for solution. This could be a serious disadvantage if the answer is not worth more than the cost to obtain it. In other words the cost of analysis using linear programming may offset the saving that may result. But with the advancement in computer technology, the solution of large linear programming problems by digital computer has not only become feasible but inexpensive as well.

2.2 FORMULATION OF LINEAR PROGRAMMING MODELS

In formulating a Linear Programming problem, the following must be taken into consideration;

a) Decision variables
These refer to any activity (product, project, etc.) that is competing with other activities for limited resources. It is important to define each variable clearly, including its dimensions so that when the solution is obtained, for example, $x_1 = 20$ we know precisely what it means.

b) The objective function.
This is sometimes called the measure of performance. It must include precisely what decisions must be made and with what objective in mind.
The objective function in a linear programming problem must be linear and the objective or a goal can be either minimization or maximization.
A properly stated objective will make the remaining steps of formulation more straight forward.

c) Constraints
These must also be linear and in addition to that they must be of $(\leq, =, \geq)$ signs. They must be in an acceptable linear inequality format, depending on whether the objective is maximization or minimization.

Example 1

$$b_1 x_1 + b_2 x_2 + b_3 x_3 + \ldots \geq d$$

Where $b_1, b_2, b_3 \ldots$ are constants and d, a non-negative constant i.e. zero or positive.

Incorrect constraint formulation can lead to solutions which are not feasible or to exclude solutions from consideration which are really feasible (and possibly optimal).

Thus,
The linear programming formulation takes the form;

Optimize $Z = f(x)$

Subject to $AX(\geq, =, \leq)b$; $b \geq 0$

For example an LP problem in n decision variables and m constraints is of the form;

Maximize $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$

Subject to the linear constraints

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1j}x_j + ... + a_{1n}x_n \le b_1$$

$$\vdots$$

$$a_{r1}x_1 + a_{r2}x_2 + ... + a_{rj}x_j + ... + a_{rn}x_n \le b_r$$

$$\vdots$$

$$a_{k1}x_1 + a_{k2}x_2 + ... + a_{kj}x_j + ... + a_{kn}x_n \le b_k$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + ... + a_{mj}x_j + ... + a_{mn}x_n \le b_m$$

$$x_i \ge 0 : i = 1, 2, ..., n$$

Note

i) The right hand side values must be Non-negative $((\ge 0))$.

ii) Optimization implies either minimization or maximization.

Example 2

Udumu Company produces two varieties of a product. Variety A has a profit per unit of $2.00 and Variety B has a profit per unit of $ 3.00.Demand for variety A is at most four units per day. Production constraints are such that at most 10 hours can be worked per day. One unit of Variety A takes one-hour to produce but one unit of product B takes two hours to produce. Ten square meters of space is available to store one day's production and one unit of Variety A requires two square meters whilst one unit of Variety B requires one square meters.

Let us now consider the above problem of the Udumu Company. The company produces two Varieties of a product, A and B. So, we define the unknown variable as follows;

Let;

x_1 =number of units of variety A produced.

x_2 =number of units of variety B produced.

These are the unknown variables which needs to be determined.

We also see that the objective of the Udumu company is to maximize profit by producing x_1 units of a variety A and x_2 units of a variety B, while taking care of the constraints given, which are; demand for variety A constraint, time constraint and space constraint. At this point we can now define clearly the constraints which must be in an acceptable linear inequality.

Constraints.

1. Demand for variety A which is at most four units per day. In inequality format we have; $x_1 \le 4$ ....(1)

2. Production time(hours) constraint
   One unit of variety A takes one-hour to produce so that the time required to
   Produce x_1 units is x_1 and one unit of product B takes two hours to produce, so that the time to produce x_2 units is $2x_2$,thus we have;
   $x_1 + 2x_2 \le 10$ ....(2)

3. Space (in meters) constraint.
   One unit of variety A requires two square meters so that x_1 units will require $2x_1$ square meters. Also, one unit of product B

requires one square meters, implies that x_2 units will occupy x_2 square meters, thus we have;

$$2x_1 + x_2 \leq 10 \text{ ....(3)}$$

In addition to that we restrict variables x_1 and x_2 to have only non-negative values. This is called the nonnegativity constraint, which the variables must satisfy. Most practical linear programming problems will have these nonnegativity restrictions on the decision variables. However, the general framework of linear programming is not restricted to nonnegative values.

As we said earlier that the objective of Udumu company is to maximize the total profit from sales. We are now assuming that there exists a perfect market for the products such that all that is produced can be sold, the total profit from sales becomes;

$$Z = 2x_1 + 3x_2$$

Thus, the linear programming model for our product mix becomes:

Find the numbers x_1, x_2 which will maximize

$$Z = 2x_1 + 3x_2$$

Subject to the constraints

$$2x_1 + x_2 \leq 10$$
$$x_1 + 2x_2 \leq 10$$
$$x_1 \leq 4$$

$$x_1 \geq 0,\ x_2 \geq 0$$

Example on clock production

Production Description: The owner of a small business manufacturing company that specializes in clocks must decide what types and quantities of output to manufacture for each day's sale. Let us assume that he manufactures only two kinds of clocks, regular

and alarm clocks from which he may select his product mix. We also assume that tomorrow's product mix can only be produced with the labour, facilities and parts currently on hand. These supplies are as follows;

Number of labour hours	1600
Number of processing hours	1800
Number of alarm assemblies	350

The resources are related to the two alternative manufactures outputs, regular clocks and alarm clocks in the following way: Each unit of regular clocks produced requires 2 hours of labour and 6 hours of processing, while each unit of alarm clock requires 4 hours of labour and 2 hours of processing.

Finally, the profit per unit of regular clocks manufactures is $3.00,while the profit per unit of alarm clocks manufactured is $ 8.00.How many of each type of clocks should the owner manufacture to maximize his profit?

Variables

We see that the company produces two types of clocks, regular clocks and alarm clocks, so we let

x_1 = number of regular clocks manufactured per day.

x_2 = number of alarm clocks manufactured per day.

These are unknown variables which need to be determined. We also note that the objective of this problem is to obtain the optimal number of regular clocks and alarm clocks so as to maximize the profit by taking care of the given constraints. Our assumption of the existing perfect market remains the same and hence we are defining the constraints again.

From the stated problem it is clear that there are three restrictions to the production process;

i) The number of labour hours available = 1600 hours
ii) The number of processing hours available = 1800 hours
iii) The number of alarm assemblies available = 350 assemblies

Let us discuss each in turn;

Labour Constraint

The number of labour hours required to produce 1 unit of regular clock is 2 hours. Thus, to produce x_1 clocks will require $2x_1$ hours. Similarly, the number of labour hours required for producing one alarm clock is 4 hours and for x_2 alarm clocks it is $4x_2$ hours. The total labour hours required to produce x_1 and x_2 clocks is $2x_1$ + $4x_2$ hours. Since only 1,600 labour hours are available, only as many clocks in total can be manufactured such that the total labour hours used in production does not exceed the hours available. Mathematically we have;

$2x_1 + 4x_2 \leq 1600$ .... (1)

Processing Constraint

To produce one unit of regular clock requires 6 hours, thus, to produce x_1 clocks will require $6x_1$ processing hours. Similarly, the number of processing hours for processing one alarm clock is 2 hours and for x_2 alarm clocks it is $2x_2$ hours. The total processing hours required to produce x_1 and x_2 clocks is $6x_1 + 2x_2$ hours, Again, since only 1800 processing hours are available we require that the total processing hours used in production do not exceed the hours available, we denote this mathematically as;

$6x_1 + 2x_2 \leq 1800$ .... (2)

Alarm assemblies' constraint

The problem states that only a supply of 350 alarm assemblies is available. Thus, the number of alarm clocks that can be produced is limited to a maximum of 350.We denote this mathematically as;

$x_2 \leq 350$

From the above constraint we can point out that it may be to the manufacturer's advantage not to use all the available alarm assemblies in this production situation and to produce less than 350 alarm clocks. In addition, we restrict the variables x_1 and x_2 to have only nonnegative values. In fact, for all Linear Programming

problems, all variables may not assume negative values. To above problem; $x_1 \geq 0,\ x_2 \geq 0$.

Objective function

We notice in the problem that $3x_1$ is the total profit from manufacturing x_1 units of regular clocks and $8x_2$ is the total profit from manufacturing x_2 units of alarm clocks.

Thus, we have the following complete formulation;

Maximize $Z = 3x_1 + 8x_2$ dollars

Subject to $2x_1 + 4x_2 \leq 1600$ labour hours

$6x_1 + 2x_2 \leq 1800$ processing hours

$x_2 \leq 350$ alarm assemblies

$x_1 \geq 0,\ x_2 \geq 0$

After this formulation a manufacturer must be able to determine the values of x_1 and x_2.To solve for x_1 and x_2 we shall see in the next section.

2.3 SOLUTION METHODS TO LINEAR PROGRAMMING PROBLEMS

In finding the solution to a Linear Programming problem we shall discuss two approaches;

1. Graphical approach (Two variable problems only)
2. The Simplex method

2.3.1 GRAPHICAL APPROACH

The method aims at getting the best possible solution. We shall discuss a graphical procedure to solve linear programming problems involving only two variables.

Consider the following already formulated linear programming problem;

Maximize $Z = x_1 + 3x_2$

Subject to $x_1 + 2x_2 \leq 10$

$x_1 \leq 5$

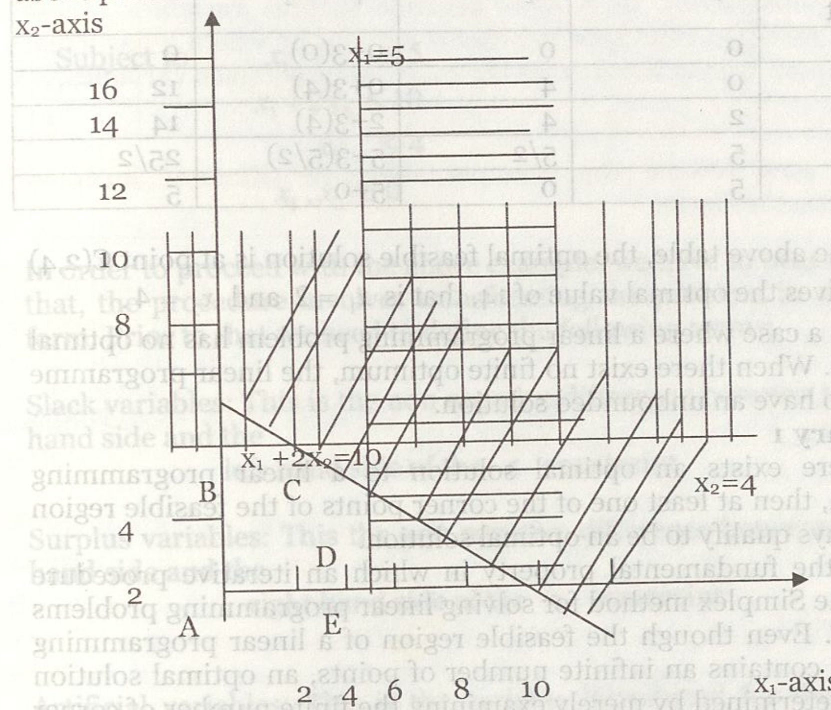
$x_2 \leq 4$

$x_1 \geq 0,\ x_2 \geq 0$

In this problem we are interested in determining the values of the variables x_1 and x_2 that will satisfy all the restrictions and give the least value for the objective function. As the first step in solving this problem, we want to identify all possible values x_1 and x_2 that are non negative and satisfy all the constraints. For example, a solution $x_1 = 3$ and $x_2 = 1$ is positive and satisfies all the constraints. Such a solution is called a feasible solution. The set of all feasible solutions is called the feasible region. Our interest will be to find the best feasible solution in the feasible region and it is called an optimal solution to the linear programming problem. In the above example, an optimal solution is a feasible solution which maximizes the objective function $x_1 + 3x_2$.The value of the objective function corresponding to an optimal solution is called the optimal value of a linear programme.

In graphical approach, every constraint is plotted, and all values of x_1, x_2 that will satisfy these constraints are identified. The nonnegativity constraints imply that all feasible values of the two variables will lie in the first quadrant. The constraint $x_1 + 2x_2 \leq 10$ requires that any feasible solution (x_1, x_2) to the problem should be on one side of the straight line $x_1 + 2x_2 = 10$ .The proper side is found by testing whether the origin satisfies the constraint or not. The line $x_1 + 2x_2 = 10$ is just plotted by computing the x_1 and x_2 intercepts (eg when x_1=0; x_2 =5 and when x_2 =0; x_1=9).This can be generalized to more that one constraint.

In this discussion, we are going to shade the unrequired region by testing the constraints with the origin that is point (0, 0) .Given the above problem we can now sketch our graph.



From the graph, corner points A,B, and E can be easily obtained. Point C and D can be obtained by solving simultaneously equations $x_1 + 2x_2 = 10$ and $x_1 = 5$ respectively. By doing so we have;

A(0,0); B(0,4); C(2,4); D(5, 5/2); E(5,0). Hence the following table.

Corner Point	x_1	x_2	x_1+3x_2	Z
A	0	0	0+3(0)	0
B	0	4	0+3(4)	12
C	2	4	2+3(4)	14
D	5	5/2	5+3(5/2)	25/2
E	5	0	5+0	5

From the above table, the optimal feasible solution is at point C(2,4) which gives the optimal value of 14, that is $x_1 = 2$ and $x_2 = 4$.

There is a case where a linear programming problem has no optimal solution. When there exist no finite optimum, the linear programme is said to have an unbounded solution.

Corollary 1

If there exists an optimal solution to a linear programming problem, then at least one of the corner points of the feasible region will always qualify to be an optimal solution.

This is the fundamental property in which an iterative procedure called the Simplex method for solving linear programming problems is based. Even though the feasible region of a linear programming problem contains an infinite number of points, an optimal solution can be determined by merely examining the finite number of corner points in the feasible region.

2.3.2 SIMPLEX METHOD

The simplex method as developed by G.B Dantzig is an iterative procedure for solving linear programming problems expressed in standard form, the simplex method requires that the constraints equations be expressed as a canonical system from which a basic feasible solution can be readily obtained.

The various steps of the simplex method can be carried out in a more compact manner by using a tableau form to represent the constraints and the objective function. In addition, by developing some simple formulas, the various calculations can be made mechanically. The tableau representation is nothing but writing the

problem in a detached coefficient form. To illustrate the method, let us consider the following example.

Maximize $Z = x_1 + 3x_2$

Subject to $x_1 \leq 5$

$x_1 + 2x_2 \leq 10$

$x_2 \leq 4$

$x_1, x_2 \geq 0$

In order to proceed with the above example, we have to bear in mind that, the procedure involves transforming the problem in standard form. Prior to that we need to define the following terms;

Slack variables: This is the non negative difference between the right hand side and the left hand side of the ≤ constraint

Surplus variables: This the non negative difference between the left hand side and the right hand side of the ≥ constraint

Artificial variables: This is the variable introduced for the limited purpose of obtaining an initial solution and are required for the constraints of ≥ type or the constraints with "=" sign.

To convert the inequality constraints into equality, we introduce slack and surplus variables. In economic terminology slack variables represent excess amount and the contribution (cost or profit coefficient) associated with them is zero. An inequality of the " less than or equal to" type is transformed into an equality by the addition of a non- negative slack variable. The slack variables can be thought of as imaginary products, each requiring for its production 1 unit of capacity from only one of the resources of zero units of capacity from the other, and each yielding a profit of zero, that is to say, a slack variable corresponds to the amount of unused capacity for the constraint to which it is added. On the other hand, an inequality of the " greater than or equal to" type is transformed into an equality by the substraction of a non- negative surplus variables. These surplus variables represent the excess amount by which a particular requirement is met. Sometimes are called negative slack variables.

Let us now rewrite the example above in standard form by employing the slacks;

Let S_1 be slack variable for constraint 1
S_2 be slack variable for constraint 2
S_3 be slack variable for constraint 3

Note also that the slack variables have no any effect in the objective function and will have a zero coefficient.
We get the following;

Maximize $Z = x_1 + 3x_2 + 0S_1 + 0S_2 + 0S_3$

Subject to

$$x_1 + S_1 = 5$$
$$x_1 + 2x_2 + 0S_2 = 10$$
$$x_2 + S_3 = 4$$
$$x_1, x_2, S_1, S_2, S_3 \geq 0$$

In tableau form we get;

C_j			1	3	0	0	0
	V_b	b_j	x_1	x_2	S_1	S_2	S_3
0	S_1	5	1	0	1	0	0
0	S_2	10	1	2	0	1	0
0	S_3	4	0	1	0	0	1
	Z_j	0	0	0	0	0	0
	Z_j-C_j		-1	-3	0	0	0

From the above table, the column V_b refers to as the basic variables column in the current basic feasible solution. The values of the basic variables are given under the column bj, the symbol C_j denotes the coefficients of the variable X_j in the objective function while in column wise, the C_j denotes the coefficients of the basic variables only.

From the above table, the basic feasible solution is immediately written as $S_1 = 5, S_2 = 10, S_3 = 4, x_1 = 0$ *and* $x_2 = 0$ and the value of the objective function is zero.
The $Z_j - C_j$ row is called an indicator row. It is used to decide whether to proceed with next iteration or not. In our case, if all $Z_j - C_j \geq 0$, then the current basic feasible solution is optimal, otherwise we continue with the next iteration.

To continue with the next iteration, we look at the most negative value in an indicator row, clearly the -3 is the most negative value, so its corresponding column is called the pivot column, and we mark this column. After we have identified a pivot column, next is to identify a pivot row, this is done by comparing the minimum ratio, the ratio of b_j column to a positive element in the pivot column. If we call elements in the table matrix a_{ij} then, the minimum ratio can be obtained by comparing all $\frac{b_j}{a_{ij}}$ where $a_{ij} \succ 0$ and a_{ij}'s are all in the pivot column. The most minimum ratio gives a pivot row and we mark it as well. If we have more than one minimum ratio, we have what is called degeneracy and we shall discuss it later. The intersection of pivot row and pivot column gives what is called a Pivot element. From the above table we have the following;

Cj			1	3	0	0	0
	V_b	b_j	x_1	x_2	S_1	S_2	S_3
0	S_1	5	1	0	1	0	0
0	S_2	10	1	2	0	1	0
0	S_3	4	0	1	0	0	1
	Z_j	0	0	0	0	0	0
	Z_j-C_j		-1	-3	0	0	0

We proceed with iterations by making use of the following formula.

$$New\ element = OE - \frac{((ACPR)x(ARPC))}{PE}$$

Where;

OE= Old element

PE= Pivot Element

ACPR= An element in the same column but pivot row

ARPC = An element in the same row but in the pivot column

x_2 enters the basis and S_3 leaves the basis and hence the following tableau;

Cj			1	3	0	0	0
	V_b	b_j	x_1	x_2	S_1	S_2	S_3
0	S_1	5	1	0	1	0	0
0	S_2	2	1	0	0	1	-2
3	X_2	4	0	1	0	0	1
	Z_j	12	0	3	0	0	3
	Z_j-C_j		-1	0	0	0	3

Cj			1	3	0	0	0
	V_b	b_j	x_1	x_2	S_1	S_2	S_3
0	S_1	3	0	0	1	-1	2
1	X_1	2	1	0	0	1	-2
3	X_2	4	0	1	0	0	1
	Z_j	14	1	3	0	1	1
	Z_j-C_j		0	0	0	0	0

Clearly, no any $Z_j - C_j \prec 0$ and as we said earlier, we have reached an optimal solution. That is $x_1 = 2,\ x_2 = 4,\ and\ S_1 = 3$ with an optimal value 14.

2.4 TIES IN THE SELECTION OF THE NON BASIC VARIABLE

The selection of the non basic variable to enter the basis is done by determining which non basic variable gives the largest per unit improvement in the objective function. In other words, in maximization case, the variable with the most negative value in the $Z_j - C_j$ row is chosen. In case there exist more than one variable with the same most negative value in the $Z_j - C_j$ row, then we have a tie for selecting the non-basic variable. The general rule is to select any one of the arbitrarily, since selecting the non-basic variable that gives the largest per unit improvement in Z need not necessarily gives the largest total improvement in Z and minimizes the number of simplex iterations.

2.5 TIES IN THE MINIMUM RATIO RULE AND DEGENERACY

While applying the minimum ratio rule it is possible for two or more constraints to give the same least ratio value. This result in a tie for selecting which basic variable should leave the basis. It may introduce further complications leading to a reduction in the efficiency of the simplex method.

Corollary 2

A basic feasible solution in which one or more basic variables are zero is called a degenerate basic feasible solution, otherwise non degenerate. A tie in the minimum ratio rule is the main cause of degeneracy in the solutions.

2.6 THE BIG M Method

For minimization problem if all $Z_j - C_j \leq 0$, then the solution is optimal. But in actual fact, the solution needs to be improved. Given a minimization problem, we proceed by introducing the artificial variables and the surplus variables and hence a Big M method.

To illustrate the method, let us consider the following Minimization problem.

Minimize $Z = 150y_1 + 20y_2 + 30y_3$

Subject to $3y_1 + 0y_2 + 8y_3 \geq 50$

$5y_1 + y_2 + 5y_3 \geq 40$

$y_1, y_2, y_3 \geq 0$

Let;

S_1 and A_1 be surplus and artificial variables for constraint 1 respectively.

S_2 and A_2 be surplus and artificial variables for constraint 2 respectively.

If we affect these, we have the following:

Minimize $Z = 150y_1 + 20y_2 + 30y_3 + 0S_1 + 0S_2 + MA_1 + MA_2$

Subject to $3y_1 + 8y_3 - S_1 + A_1 = 50$

$5y_1 + y_2 + 5y_3 - S_2 + A_2 = 40$

$y_1, y_2, y_3, S_1, S_2, A_1, A_2 \geq 0$

In tableau form we have;

c_j			150	20	30	0	0	M	M
	V_b	b_j	y_1	y_2	y_3	S_1	S_2	A_1	A_2
M	A_1	50	3	0	8	-1	0	1	0
M	A_2	40	5	1	5	0	-1	0	1
	Z_j	90M	8M	M	13M	-M	-M	M	M
	Z_j-C_j		8M-150	M-20	13M-30	-M	-M	0	0

The rationale behind here is that on initial table when we didn't introduce the big M method together with the artificial variable allows us to get the initial solution which is not optimal. The criterion here is that all artificial variables must go out the basis first. We are now looking at the indicator row, the $Z_j - C_j$ if all $Z_j - C_j \prec 0$,if no, we go for the most positive value ,and we find that 13M-30 is the most positive value in an indicator row, just as we did in the previous example, the same procedures applies here.

2nd iteration.

C_j			150	20	30	0	0	M	M
	V_b	b_j	y_1	y_2	y_3	S_1	S_2	A_1	A_2
30	y_3	50/8	3/8	0	1	-1/8	0	1/8	0
M	A_2	35/4	25/8	1	0	5/8	-1	-5/8	1
Z_j		1500/8+35M/4	90/8+25M/8	M	30	-30/8+5M/8	-M	30/8-5M/8	M
Z_j-C_j			25/8M-555/4	M-20	0	5M/8-30/8	-M	30/8-13M/8	0

3rd iteration

C_j			150	20	30	0	0	M	M
	V_b	b_j	y_1	y_2	y_3	S_1	S_2	A_1	A_2
30	y_3	50/8	3/8	0	1	-1/8	0	1/8	0
20	y_2	35/4	25/8	1	0	5/8	-1	-5/8	1
Z_j		2900/8	590/8	20	30	70/8	-20	-70/8	20
Z_j-C_j			305/4	0	0	70/8	-20	-70/8-M	20-M

4th iteration

C_j			150	20	30	0	0	M	M
	V_b	b_j	y_1	y_2	y_3	S_1	S_2	A_1	A_2
30	y_3	26/5	0	-3/25	1	-1/5	3/25	1/5	-3/25
150	y_1	14/5	1	8/25	0	5/25	-8/25	-5/25	8/25
Z_j		2880/5	150	222/5	30	24	- 225/5	-24	222/5
Z_j-C_j			0	122/5	0	24	-222/5	-24-M	222/5-M

5th iteration

C_j			150	20	30	0	0	M	M
	V_b	b_j	y_1	y_2	y_3	S_1	S_2	A_1	A_2
30	y_3	25/4	3/8	0	1	-1/8	0	1/8	0
20	y_2	35/4	25/8	1	0	5/8	-1	-5/8	1
Z_j		1450/4	590/8	20	30	70/8	-20	-70/8	20
Z_j-C_j			-305/4	0	0	70/8	-20	-70/8-M	20-M

6th iteration

C_j			150	20	30	0	0	M	M
	V_b	b_j	y_1	y_2	y_3	S_1	S_2	A_1	A_2
30	y_3	8	1	1/5	1	0	-1/5	0	1/5
0	S_1	14	5	8/5	0	1	-8/5	-1	8/5
Z_j		240	30	30/5	30	0	-30/5	0	30/5
Z_j-C_j			-120	-14	0	0	-30/5	-M	30/5-M

Since all $Z_j - C_j \prec 0$ then the current solution is optimal .That is y_3=8, S_1=14 and the optimal value is 240.

2.7 THE TWO PHASE SIMPLEX METHOD

In here the linear programming problem is solved in two phases.

2.7.1 Phase 1

It involves finding an initial basic feasible solution to the original problem. In other words the removal of the artificial variables is taken up first. For this, an artificial objective function is created which is the sum of all the artificial variables. The artificial problem is then minimized using the simplex method. If the minimum value of the artificial problem is zero, then all the artificial variables have been reduced to zero and we have a basic feasible solution to the original problem.

If we fail to remove all the artificial variables in this phase, then the solution is infeasible and we terminate.

2.7.2 Phase II

In this phase, the basic feasible solution found at the end of phase I is optimized with respect to the original objective function. In other words, the final tableau of phase one becomes the initial tableau for phase II after changing the objective function and at this point we omit the columns of the artificial variables. The simplex method is once again applied to determine the optimal solution.

Corollary 3

Given a minimization LP problem, the problem can be changed to maximization by multiplying the objective function with -1 without affecting the constraints.

We illustrate the Two-Phase approach using the following linear programming problem.

Minimize $Z = -3x_1 - 2x_2 - 4x_3$

Subject to $x_1 + x_2 + x_3 \geq 12$

$4x_1 - x_2 \geq 6$

$x_1, x_2, x_3 \geq 0$

Let;

x_4 = surplus variable in the first constraint

x_5 = artificial variable in the first constraint.

x_6 = surplus variable in the second constraint

x_7 = artificial variable in the second constraint

The connection with this will be as follows;

Phase I

Maximize $Z = 0x_1 + 0x_2 + 0x_3 + 0x_4 - 1x_5 + 0x_6 - 1x_7$

Subject to $x_1 + x_2 + x_3 - x_4 + x_5 = 12$

$4x_1 - x_2 - x_6 + x_7 = 6$

$x_i 's \geq 0;\ i = 1,2,\ldots,7$

Tableau form.

C_j			0	0	0	0	-1	0	-1
	V_b	b_j	X_1	X_2	X_3	X_4	X_5	X_6	X_7
-1	X_5	12	1	1	1	-1	1	0	0
-1	X_7	6	4	-1	0	0	0	-1	1
	Z_j	18	-5	0	-1	1	-1	1	-1
	Z_j-C_j		-5	0	-1	1	0	1	0

After two iterations we have the following solution,

C_j			0	0	0	0	-1	0	-1
			x_1	x_2	x_3	x_4	x_5	x_6	x_7
	V_b	b_j							
0	X_2	42/5	0	1	4/5	-4/5	4/5	1/5	-1/5
0	X_1	18/5	1	0	1/5	-1/5	1/5	-1/5	1/5
	Z_j	0	0	0	0	0	0	0	0
	Z_j-C_j		0	0	0	0	1	0	1

and we note that artificial variables are all driven out of the basis and that the objective function value is zero. Thus we proceed with the second phase.

Phase II

After eliminating column for artificial variables the phase I solution becomes our initial basic feasible solution. So we have;

Initial Tableau

C_j			-3	-2	-4	0	0
	V_b	b_j	x_1	x_2	X_3	x_4	x_5
-2	x_2	42/5	0	1	4/5	-4/5	1/5
-3	x_1	18/5	1	0	1/5	-1/5	-1/5
	Z_j	-138/5	-3	-2	-11/5	11/5	1/5
	Z_j-C_j		0	0	9/5	11/5	1/5

Since all $Z_j - C_j \geq 0$ and that the basis contains no artificial variable, then the current solution is optimal.

$$X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (\frac{18}{5}, \frac{42}{5}, 0,0,0,0,0)$$

We note that $Z = -Z = -\left(\frac{-138}{5}\right) = 138/5$ which is our minimum value.

COMPARISON BETWEEN BIG M SIMPLEX METHOD AND THE TWO-PHASE APPROACH

The basic approach to both methods is the same. Both add artificial variables to get the initial canonical system and then drive them to zero as soon as possible.

1. The number of iterations are the same.
2. The Big M method solves an LP in one pass while the Two Phase method solves it in two stages as two linear programs.

2.8 SENSITIVITY ANALYSIS IN LINEAR PROGRAMMING

In all linear programming models the coefficients of the objective function and the constraints are supplied as input data or as parameters to the model. The optimal solution obtained from the simplex method is based on the values of these coefficients. The values of these coefficients are known with absolute certainty, because many of these are functions of uncontrollable parameters.

For example the future demand, the cost of raw materials can not be predicted with complete accuracy before the problem is solved. In this case the solution of a practical problem is not complete with a mere determination of the optimal solution. Changing any coefficient in the objective function or any value in the right hand side of the $(\geq, =, \leq)$ constraint will in turn affect the optimal solution found earlier. Analyzing the change in optimal solution with respect to changes in the input (data) coefficient is known as Sensitivity Analysis.

To illustrate the Sensitivity analysis, let us consider the following example.

$$\text{Maximize } Z = 50x_1 + 40x_2$$

$$\text{Subject to} \quad 3x_1 + 5x_2 \le 150$$

$$x_2 \le 20$$

$$8x_1 + 5x_2 \le 300$$

$$x_1, x_2 \ge 0$$

The final tableau to the above problem is;

c_j		50	40	0	0	0
	V_b b_j	x_1	X_2	x_3	x_4	x_5
40	x_2 12	0	1	8/25	0	-3/25
0	x_4 8	0	0	-8/25	1	3/25
50	x_1 30	1	0	-5/25	0	5/25
Z_j	1980	50	40	70/25	0	130/250
Z_j-C_j		0	0	70/25	0	130/25

Case I

♦ Changing the objective function coefficient of a non basic variable.

At optimal solution the indicator row obeys the criteria that $Z_j - C_j \ge 0$ which implies that $C_j \le Z_j$

Consider the non-basic variable X_3;

$$Z_3 - C_3 = (40 \quad 0 \quad 50)\begin{pmatrix} 8/25 \\ -8/25 \\ -5/25 \end{pmatrix} - C_3$$

Z_3 here is represented in a matrix form.

$320/25 + 0 + -250/25 - C_3$

$70/25 - C_3$; from Z_j-$C_j \ge 0$

$70/25 - C_3 \ge 0$

$C_3 \le 70/25$

As long as $C_3 \le 70/25$, the solution will remain optimal.

On the other hand , if $C_3 = 75/25$, when substituting to the expression Z_j-$C_j \ge 0$

$70/25 - 75/25 \ge 0$

$C_3 \le Z_3$

$75/25 \le 70/25$

$3 \le 2.8$ which is not true and in this case we have $Z_j \le Cj$ which needs improvement.

Case II

♦ The change in profit value C_1 say, of unit of product 1 has an effect on the product mix;

i) If the profit is decreased, the product one may not be profitable as to be included in the optimal solution.

ii) If the profit is increased, it may be so profitable that only this product can be in the optimal solution.
Consequently, such a profit will have a lower and upper limit for the current optimal solution. Changing C_1 causes V_b column to change, but will have no effect in the indicator row (Z_j-C_j).
The indicator value corresponding to the non-basic variables can be expressed in terms of C_1 as follows;

Example 3

$$Z_3 - C_3 = \begin{pmatrix}40 & 0 & c_1\end{pmatrix}\begin{pmatrix}8/25 \\ -8/25 \\ 5/25\end{pmatrix} = \frac{320}{25} - 0 + \frac{5c_1}{25}$$

$$= 64/5 + C_1/5$$

$$Z_5 - C_5 = \begin{pmatrix}40 & 0 & c_1\end{pmatrix}\begin{pmatrix}-3/25 \\ 3/25 \\ 5/25\end{pmatrix} = \frac{-120}{25} + \frac{5C_1}{25}$$

$$= -24/5 + C_1/5$$

Since at optimal $Z_j - C_j \geq 0$; then we have also

$64/5 + C_1/5 \geq 0$ which gives $C_1 \leq 64$
$-24/5 + C_1/5 \geq 0$ which gives $C_1 \geq 24$

Thus $24 \leq C_1 \leq 64$

Case III

♦ Changing the Right hand side values.
The process of determining the range of feasibility for the right hand side of the less than or equal to constraint, I is as follows;

Define Δb_i to be the change of bi of constraint i

$$\begin{bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ \cdot \\ b_m \end{bmatrix}$$ = current solution (i.e. the last column of the final simplex tableau)

$$\begin{bmatrix} a_{ij} \\ a_{2j} \\ \cdot \\ \cdot \\ \cdot \\ a_{nm} \end{bmatrix}$$ =column of the final simplex tableau corresponding to the slack, surplus variables associated with constraint i.

We also know that the basic solution is feasible when $b_i \geq 0$ for all i.

The range of b_i is determined by solving the following simultaneous inequalities .This is for a maximization problem.
The used simultaneously inequalities are;

$$\begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{bmatrix} + \Delta b_i \begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ . \\ a_{mj} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} \quad ....1$$

Considering our example we had; for bi

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \\ 30 \end{bmatrix}; \quad \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} = \begin{bmatrix} 8/25 \\ -8/25 \\ -5/25 \end{bmatrix}$$

Then we have in (1)

$$\begin{bmatrix} 12 \\ 8 \\ 30 \end{bmatrix} + \Delta b_i \begin{bmatrix} 8/25 \\ -8/25 \\ -5/25 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$12 + \Delta b_i\, 8/25 \geq 0$

$\Rightarrow \quad \Delta b_i \geq -37.5$ .... (i)

$8 + -\Delta b_i\, 8/25 \geq 0$

$\Rightarrow \quad \Delta b_i \leq 25$ .... (ii)

$30 + -\Delta b_i\, 5/25 \geq 150$

$\Rightarrow \quad \Delta b_i \leq 50$ ....(iii)

Upon comparison of the three values of Δb_i, we see that $-37.5 \leq \Delta b_i \leq 25$

Recall that the initial amount available for constraint 1 that is $b_1=150$, the range of b_1 is obtained by looking into the limits of $b + \Delta b_i$ that is;

$150-37.5 \leq b + \Delta b_i \leq 25+150$

$112.5 \leq b + \Delta b_i \leq 175$

So long as $b_1 \in [112.5, 175]$ the current optimal solution will remain feasible. The range above allows us to vary resources to that range.

For a minimization case we use the following formula;

$$\begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{bmatrix} - \Delta b_i \begin{bmatrix} a_{1j} \\ a_{2j} \\ . \\ . \\ . \\ a_{mj} \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix}$$

For equality constraint the procedure is complex and we will not cover it here.

2.9 DUALITY THEOREM

The theory is one of the most important concepts in linear programming. The basic idea here is that every linear program has an associate linear program called its dual such that a solution to one gives a solution to the other.

Linear programming theory states that each problem that we formulate is in reality two problems which we shall call the primal problem and the dual problem. Although we may formulate a specific linear programming problem and solve it, there exists another linear programming problem which uses the same data but is called the dual problem.

Relationships between Primal and Dual:

1. The dual of dual is a primal problem
2. If one problem has an optimal solution, the other has an optimal solution too.
3. If the primal is unbounded, the dual is infeasible
4. If the primal is infeasible, the dual may be either unbounded or infeasible.

Suppose Z_k-C_k is the indicator value to the original basic primal variable in the kth primal constraint, the k[th] dual variable value is obtained by using the relation;

$$Y_k = |(Z_k - C_k) + C_k|;$$

Similary,the primal decision variable from the dual tableau is given by

$$X_k = |(W_k - b_k) + b_k|;$$

Where, $W_k - b_k$ is the indicator row value under the original basic variable for the k[th] constraint.
For example;
The final tableau to the primal problem

$$\text{Max } Z = 8x_1 + 4x_2$$

$$\text{S/t} \quad x_1 + x_2 \leq 10$$
$$5x_1 + x_2 \leq 15$$
$$x_1, x_2 \geq 0$$

Is given below;

C_j			8	4	0	0
	V_b	b_j	x_1	x_2	x_3	x_4
4	x_2	35/4	0	1	5/4	-1/4
8	x_1	5/4	1	0	-1/4	1/4
	Z_j	45	8	4	3	1
	Z_j-C_j		0	0	3	1

The variables x_3 and x_4 are slacks for the first and second constraints respectively.
We also note that x_3 correspond to the first constraint in the original problem, thus y_1 in the dual, also x_4 corresponds to the second constraint hence y_2 in the dual .If we affect these results we get;

$$Y_1 = |(Z_3 - C_3) + C_3| = |(3-0)+0| = 3$$

$$Y_2 = |(Z_4 - C_4) + C_4| = |(1-0)+0| = 1$$

2.9.1 Formulation of a Dual problem

There are three ways in formulating the dual problem,

1. From canonical primal.
2. From general primal
3. Standard form primal

Any primal problem can be modified to one of the three forms above.
Let us consider the
Canonical primal form;

- For a maximization primal objective, the dual is minimization.
- Maximization primal must have all ≤ constraints
- The X and Y decision variables for the primal and dual problem respectively are non-negative.
- To every constraint in one problem, there exists one variable in the other and vice-versa
- The right hand side elements of one problem are the respective coefficients on the objective function of the other and vice-versa.
- The coefficient matrix of one problem is the transpose to the other.

In general for any;

Max Z= CX
S/t AX ≤ b Primal problem

X ≥ 0

There exists another problem

Min W = $B^T y$

S/t $A^T Y \geq CT$
$Y \geq 0$.
Called the dual problem.

From the General form

In this approach, the objective may be maximization or minimization. The constraints can be ≤, ≥ or = type, and variables can be restricted or unrestricted.

- When the primal objective is maximization, that of the dual is minimization and vice-versa.

- For maximization objective, the constraints must be ≤ and for minimization objective the constraints must be ≥.
- Every constraint of one problem results in one variable in the other and vice-versa
- The right hand side vector of one problem is the coefficients of the variables in the objective of the other and vice –versa.
- The coefficient matrix of one problem is the transpose of the other and vice-versa.
- If the variable in one problem is unrestricted, the resulting constraint in the other is equality (=) and vice-versa.

Equality Constraints

Consider the following equality constraint

$$g_i(x) = b_i$$

Where $g_i(x)$ is a linear combination of decision variables, we write the above as;

$b_i \leq g_i(x) \leq b_i$ or $g_i(x) \leq b_i$ and $g_i(x) \leq b_i$

this means that an equality constraint can be broken down into two inequality constraints.

Example 4

Find the dual for

$$\text{Max } Z = 4x_1 + x_2 + 7x_3$$

$$\begin{aligned} \text{s/t} \quad & x_1 + x_2 + x_3 = 10 \\ & 5x_1 - x_2 + x_3 \geq 10 \\ & x_1 + 7x_2 - 3x_3 \leq 10 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

Solution

We create two constraints from the equality constraint

$$10 \le x_1 + x_2 + x_3 \le 10$$

$$\Rightarrow \quad x_1 + x_2 + x_3 \le 10 \text{ or } \quad x_1 + x_2 + x_3 \ge 10$$

To conform to the canonical form, all constraints must be of ≤ type

Max $Z = 4x_1 + x_2 + 7x_3$

s/t $\quad x_1 + x_2 + x_3 \le 10$

$$-x_1 - x_2 - x_3 \le -10$$

$$-5x_1 + x_2 - x_3 \le -12$$

$$x_1 + 7x_2 - 3x_3 \le 4$$

$$x_1, x_2, x_3 \ge 0$$

The dual for it will take the form;

Min W = $b^T y$

s/t $\quad A^T Y \ge C^T$

$\quad Y \ge 0$

Where;

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ -5 & 1 & -1 \\ 1 & 7 & -3 \end{pmatrix}; \quad A^T = \begin{pmatrix} 1 & -1 & -5 & 1 \\ 1 & -1 & 1 & 7 \\ 1 & -1 & -1 & -3 \end{pmatrix}$$

$$b = \begin{pmatrix} 10 \\ -10 \\ -12 \\ 4 \end{pmatrix}; \quad b^T = \begin{pmatrix} 10 & -10 & -12 & 4 \end{pmatrix}$$

$$C = \begin{pmatrix} 4 & 1 & 7 \end{pmatrix} \text{ and } C^T = \begin{pmatrix} 4 \\ 1 \\ 7 \end{pmatrix}$$

So that;

Min $W = 10y_1 - 10y_2 - 12y_3 + 4y_4$

Such that $\quad y_1 - y_2 - 5y_3 + y_4 \ge 4$

$$y_1 - y_2 + y_3 + 7y_4 \ge 1$$

$$y_1 - y_2 - y_3 - 3y_4 \ge 7$$

$$y_i \ge 0; i = 1, \ldots, 4$$

If we let $y_1 - y_2 = y_{12}$, then we have;

Min $W = 10y_{12} - 12y_3 + 4y_4$

Such that $\quad y_{12} - 5y_3 + y_4 \ge 4$

$$y_{12} - y_3 + 7y_4 \ge 1$$

$$y_{12} - y_3 - 3y_4 \ge 7$$

$y_1, y_4 \ge 0$; y_{12} Unrestricted

Example 5

Write the dual for

Min $Z = 4x_1 + 2x_2 - x_3$

s/t $\quad x_1 + x_2 + x_3 = 20$

$2x_1 - x_2 \geq 6$

$x_3 \leq 4$

$x_1, x_2 \geq 0$, x_3 unrestricted

Solution

Constraint one can be rewritten as;

$20 \leq x_1 + x_2 + x_3 \leq 20$

$\Rightarrow \; x_1 + x_2 + x_3 \leq 20$ and $x_1 + x_2 + x_3 \leq 20$

to conform with canonical form all constraints must be of $\geq$ type.

Min $Z = 4x_1 + 2x_2 - x_3$

s/t $\quad x_1 + x_2 + x_3 \geq 20$

$-x_1 - x_2 - x_3 \geq -20$

$2x_1 - x_2 \geq 20$

$x_1, x_2 \geq 0$, x_3 Unrestricted

The dual formulation will be;

Max $W = 20y_1 - 20y_2 + 6y_3 - 4y_4$

s/t $\quad y_1 - y_2 + 2y_3 + 0y_4 \leq 4$

$y_1 - y_2 - y_3 + 0y_4 \leq 2$

$y_1 - y_2 + 0y_3 - y_4 \leq -1$

this can be rewritten as

Max $W = 20y_{12} + 6y_3 - 4y_4$

s/t $\quad y_{12} + 2y_3 \leq 4$

$y_{12} - y_3 \leq 2$

$y_{12} - y_4 \leq -1$

$y_{12} \geq 0; y_3$ Unrestricted.

2.10 ECONOMIC INTERPRETATION OF THE DUAL VARIABLES

Consider the following example;

Siyawezi,a third year graduate in one University accepted a job with Maliyatabu firm. The job was a combination of purchasing agent, marketing officer and trouble shooter.Maliyatabu performed the basic task of serving as a middleman for the farmers in a Ogopatapeli region in Bongoland.What Maliyatabu did was to store tobacco, maize and cotton for the farmers and to serve as their agent in selling them.They made their money by taking a percentage of the amount received.Siyawezi has received a bachelor degree in business. She built on her own time, a mathematical model of the firm's operations which was actually derived from the following primal problem;

Optimize $W = 10000y_1 + 600y_2 + 130y_3 + 80y_4 + 200y_5$

Such that $\quad 40y_1 + 3y_2 + y_3 \geq 10$

$50y_1 + 6y_2 + y_4 \geq 18$

$60y_1 + 2y_2 + y_5 \geq 7$

$y_i's \geq 0;\; i = 1,2,3,4,5$

and she solved the following dual problem;

Optimize $Z = 10x_1 + 18x_2 + 7x_3$

Such that $\quad 40x_1 + 50x_2 + 60x_3 \leq 10000$ .... (1)

$3x_1 + 6x_2 + 2x_3 \leq 600$ .... (2)

$x_1 \leq 130$ .... (3)

$x_2 \leq 80$ .... (4)

$x_3 \leq 200$ .... (5)

where;

1= constraint of Maliyatabu floor space limitation of 10,000 m³
2= limitation constraint of person-hours (600 person hours in total) over a period of
interest.
3 through 5 reflect the Ogopatapeli region predicted output for the three crops.

Let x_1 = tons of maize stored
x_2 = tons of tobacco stored
x_3 = tons of cotton stored

On solving the problem, Siyawezi came out with this optimal tableau.

C_j			10	18	7	0	0	0	0	0
	V_b	b_j	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
7	x_3	915/13	0	0	1	3/130	-5/26	-9/26	0	0
10	x_1	130	1	0	0	0	0	1	0	0
0	x_7	890/13	0	0	0	1/130	-9/39	5/13	1	0
18	x_2	150/13	0	1	0	-1/130	9/39	-5/13	0	0
0	x_8	1685/13	0	0	0	-3/130	5/26	9/26	0	0
	Z_j		0	0	0	3/130	73/26	17/26	0	0

From this tableau we have the optimal solution;

$x_1 = 130$, $x_2 = \dfrac{150}{13}$ and $x_3 = \dfrac{915}{13}$ which results into $Z = \dfrac{26005}{13}$ maximum profit.

Since x_4 to x_8 represent slack variables to the primal basic solution, the corresponding dual variables are obtained by using the relation.

$Y_k = |(Z_k - C_k) + C_k|$ thus;

$Y_1 = |(3/130 - 0) + 0| = 3/130$ per unit contribution of space.

$Y_2 = |(73/26 - 0) + 0| = 73/26$ per unit contribution of labour.

$Y_3 = |(17/26 - 0) + 0| = 17/26$ per unit contribution of maize.

$Y_4 = |(0 - 0) + 0| = 0$ per unit contribution of tobacco.

$Y_5 = |(0 - 0) + 0| = 0$ per unit contribution of cotton.

This implies that addition of more tobacco or cotton will not increase Maliyatabu firm profit. Increase in profit is greatly obtained by increasing person-hours (the firm can arrange for overtime for Siyawezi).As regards to maize and space, the firm can look for space to rent so as to store more maize to be bought. The price to pay for extra labour, maize and space must be less than their contribution to the total profit. But time should be spent in checking the validity of estimates of profit market availability. Can Maliyatabu just go on increasing maize stored, space or labor before the contribution change? These are just some of the aspects to be considered.

2.11 RELATIONSHIP BETWEEN DUAL AND PRIMAL TABLEAU

Consider the following example;

Example 6

The final tableau of the dual problem of the primal problem is;

Minimize $Z = 2x_1 + 4x_2 + x_3$

Such that $x_1 + 2x_2 - 2x_3 \leq 5$

$2x_1 - x_2 + 2x_3 = 2$

$x_1 + 2x_2 + 2x_3 \geq 1$

$x_1, x_2, x_3 \geq 0$

C_j			-5	2	1	0	0	0
	V_b	b_j	y_1	y_2	y_3	y_4	y_5	y_6
0	y_4	1	-2	0	-3	1	0	-1
0	y_5	9/2	-3/2	0	3	0	1	1/2
0	y_2	1/2	1/2	1	1	0	0	1/2
	Z_j		6	0	1	0	0	1

Where y_4, y_5 and y_6 are the dual slack variables. Find the optimal solution to the given problem.

Solution

In this case the relation

$X_k = |(W_k - C_k) + C_k|$ is to be used

x_1 correspond to y_4 slack variable in the dual basic

x_2 correspond to y_5 slack variable in the dual basic

x_3 correspond to y_6 slack variable in the dual basic

Therefore;

$X_1 = |(W_4 - C_4) + C_4|$

$= |(0-0)+0| = 0$

$X_2 = |(W_5 - C_5) + C_5| = 0$

$X_3 = |(W_6 - C_6) + C_6|$

$= |(1-0)+0| = 1.$

and so Z=W=1.

Exercise

1. A pension – fund manager is considering investing in two shares, X and Y. it is estimated that share X will earn a dividend of 12 percent per annum and share Y 4 percent per annum. Growth in the market value in one year of share X will be 10 cent per shilling invested and in Y 40 cents per shilling invested. He requires to invest the minimum total sum which will give dividend income of at least Tshs.600,000 per annum; and growth in one year of at least Tshs.1,000,000 in the initial investment.
   (i) State the mathematical formulation of the problem.
   (ii) State the dual of (i).

2. Breeding manufacturing company produces two types of hydraulic pumps namely the standard and oversize pumps. The manufacturing process associated with production of pumps involves three activities; Assembly, Painting and Testing (Quality control) Resources requirements for

assembly, painting and testing of the pumps are given in the table below;

Manufacturing Requirements (hrs)

Type	Assembly time	Paining time	Testing time
Standard	3.6	1.6	0.6
Oversize	4.8	1.8	0.6

Profit from sale of standard pump is USD 50, from oversize is USD 75. There are 4800 hrs of assembly time, 1980 hrs painting time and 900 hrs of test time available per week. Prior sales experiences indicate that the company can expect to sell at least 300 standard pumps and 180 oversize pumps per week. Breeding would like to determine the quantity of each type of pumps per week. Breeding would like to determine the quantity of each type of pump to produce weekly to maximize profit;

(a) Formulate a linear programming problem
(b) Process it ready for simplex method
(c) Present an initial tableau.

3. Write the dual of the following:

Maximize $20x_1 + 25x_2$

Such that

$$40x_1 + 30x_2 \leq 850$$
$$5x_1 + 10x_2 \leq 200$$
$$x_1, x_2 \geq 0$$

4. In a 10 – block city area, there are currently only two bus routes, one going in either direction along the same street. The city manager wants to maximize the bus traveling population by adding alternate routes along the present 10 bus routes. The routes are all straight, there are a number of bus stops on each route, and the capacity of each bus is limited. Is this an LP problem? Why?

5. when is duality required to solve an LP problem?

6. A manufacturer makes two type of products, I & II. One unit of product requires 2 hrs on machine A, 1 hr on machine B and 6 hrs on machine C; one unit of product II requires respectively, 2hrs, 5hrs and 2 hrs on machines A, B and C. Machine A is available for 24hrs, B for 44 hrs and C for 60hrs. Profit per unit of product I is USD 6 and for product II is USD 9. Given that the machines are available when required, how many units of each product should be made in order to maximize profit? (Use simplex method).

7. Use graphical method to solve the following linear programming problem;

Minimize $Z = 40x_1 + 36x_2$

Such that

$$x_1 \leq 8$$
$$x_2 \leq 10$$
$$5x_1 + 3x_2 \geq 45$$
$$x_1, x_2 \geq 0$$

CHAPTER THREE

TRANSPORTATION AND ASSIGNMENT PROBLEMS

3.0 Introduction

In the previous section we saw broad class of problems that can be formulated and solved using the linear programming framework. However, in most of those problems, the simplex method, although powerful enough to solve all the problems, is not the most efficient solution technique to other problems. One such special type of problem is known as the transportation problem. The special procedure for solving this class of problems is applicable to many situations that have the special transportation type structure in their formulation. Another special type of problem is the assignment problem, which is in reality a special case of the transportation problem but with a more refined structure.

3.1 Statement of the problem

A manufacturer produces a single product at I factories (sources).These goods must then be shipped to N storage facilities (destination). Each source has a known availability supply and each destination has a known required demand. If the cost of shipping a single unit of the product from each source to each destination is known, our objective is to determine a shipping schedule from sources to destinations such that the total shipping cost is as small as possible.

3.2 Definition

Transportation problems deal with situations where the product at sources is required to be shipped to various destinations such that the transportation cost is minimized.

Consider the distribution problem where a certain item is produced in factory i; $i = 1,2,....,m$ with capacities S_i, and it is required in regions j, $j = 1,2,....,n$, requiring d_j,

Let X_{ij} denote the amount of item transported from factory i to region j and also that

$$\sum_{i}^{m} x_{ij} = \text{Total amount of the item from all factories to region } j.$$

$$\sum_{j}^{n} x_{ij} = \text{Total amount of item from factory } i \text{ to all regions}$$

Then $\sum_{i}^{m} x_{ij} \geq d_j$ and $\sum_{j}^{n} x_{ij} \leq S_i$

Let d_{ij} = distance from factory i to region j

If we hire transport from one firm charging C Tshs per tone per km, Then the total transportation cost will be ;

$$\sum_{i}^{m}\sum_{j}^{n} Cd_{ij}X_{ij} \text{ .... (1)}$$

On the other hand if the cost of hiring a transport from factory i to region j per tone of the item is irrespective of distance, then the total transportation cost is;

$$\sum_{i}^{m}\sum_{j}^{n} C_{ij}X_{ij} \quad ....(2)$$

where C_{ij} is the charge of transporting a unit of the item from factory i to region j.

If we want to minimize the total transportation cost for alternative (2), the formulation can be stated as;

$$\text{Min} \quad \sum_{i}^{m}\sum_{j}^{n} C_{ij}X_{ij}$$

Subject to $\sum_{j}^{n} x_{ij} \leq S_i \;;\; i = 1,2,...m$

$$\sum_{i}^{m} x_{ij} \geq d_i \;;\; j = 1,2,...n$$

$$X_{ij} \geq 0$$

For the problem to be feasible we need $\sum_i S_i \geq \sum_j d_j$

3.3 VARIATIONS OF TRANSPORTATION PROBLEM

1. Total supply exceeds the total demand

   The transportation problem is balanced by creating a dummy demand point which has a demand equal to the amount of excess supply ($\sum_i S_i - \sum_j d_j$).This is unused supply, since is not real shipment and are given zero transportation cost.

2. Total supply is less than total demand
   If this happens, we have infeasible transportation problem. To balance the problem we create a dummy supply point having a supply of $\sum d_j - \sum S_i$.
   A penalty cost of zero is associated to all transported amounts from the dummy supply point.

3. Unacceptable routes
   It may happen that not all demand points receive items from every supply point. In this case we simply drop the corresponding route from the network. Usually the dropped arc capacity is included as a constraint, $X_{ij} = 0$.

3.4 SOLUTION METHODS TO TRANSPORTATION PROBLEMS

To solve the TP, it is necessary first of all to balance it by adding a dummy supply or demand point. Then, as we did in LP, we need to find an initial basic feasible solution. This initial basic feasible solution is then improved to get an optimal if it exists. Initial basic solution for a balanced TP can be found by using any of the following methods;

1. Minimum Cost
2. Northwest Corner
3. Vogel's Approximation method (VAM)

3.4.1 MINIMUM COST METHOD

In this method the variables of minimum cost enter the basis first. Thereafter the satisfied row or column is crossed out and necessary adjustment is made. We can illustrate the method by considering the following example;

To / From	1	2	3	4	SUPPLIES
A	**15**	**18**	**19**	**13**	50
B	**20**	**14**	**15**	**17**	30
C	**25**	**12**	**17**	**22**	70
DEMANDS	30	60	20	40	

Solution

We see that $\sum S_i = \sum d_j$ = 150 and so we can proceed.

From the problem above, the variable X_{32} where the subscript denotes row and column respectively has the minimum cost, thus

X_{32} = minimum $\{s_3, d_2\}$

= minimum $\{70,60\}$ =60

Cross out column 2 and adjust S_3 to S^*_3 =10,if you do that we get;

To / From	1	2	3	4	SUPPLIES
A	**15**	**18**	**19**	**13**	50
B	**20**	**14**	**15**	**17**	30
C	**25**	(60) **12**	**17**	**22**	~~70~~ 10
DEMANDS	30	60	20	40	

The next minimum cost variable is x_{14}

x_{14} = minimum $\{s_1, d_4\}$

= minimum $\{50,40\}$ =40 so we cross column four and adjust s1 to s_1^*=10, so we get;

To / From	1	2	3	4	SUPPLIES
A	**15**	**18**	**19**	(40) **13**	~~50~~ 10
B	**20**	**14**	**15**	**17**	30
C	**25**	(60) **12**	**17**	**22**	~~70~~ 10
DEMANDS	30	60	20	40	

Next minimum again is at x_{11} and x_{23} .

Consider the x_{23} variable which has least indices compared to x_{11}

x_{23} = min $\{s_2, d_3\}$

= min $\{30,20\}$ =20

cross the third column and adjust s_2 to s_2^* =10 proceed with this fashion we have the following;

From \ To	1	2	3	4	SUPPLIES
A	15 (10)	18	19	(41)	~~50~~ ~~10~~ 0
B	20 (10)	14	15 (20)	17	~~30~~ ~~10~~ 0
C	25 (10)	(60) 12	17	22	~~70~~ ~~10~~ 0
DEMANDS	~~30~~ 0	60	20	40	

So that the transportation cost associated with this plan is;
(10x 15) +(10x20) +(10x25) + (60x12) + (20x15) + (40x13)= 2140.

3.4.2 NORTHWEST CORNER METHOD

Let us consider the previous example to illustrate the method.

From \ To	1	2	3	4	SUPPLIES
A	15	18	19	13	50
B	20	14	15	17	30
C	25	12	17	22	70
DEMANDS	30	60	20	40	

The most N-W Corner variable is x_{11}.In this method we also compare the minimum of either supply or demand.

$$x_{11} = \min \{s_1, d_1\}$$
$$= \min \{50,30\} = 30$$

cross first column and adjust s_1 to $s_1^* = 20$

From \ To	1	2	3	4	SUPPLIES
A	15 (30)	18	19	13	50 20
B	20	14	15	17	30
C	25	12	17	22	70
DEMANDS	30	60	20	40	

Then N-W Corner variable is x_{22}.

$$x_{22} = \min \{s_2, d_2\}$$
$$= \min \{30,60\} = 30$$

cross the second column and adjust d_2 to $d_2^* = 30$

To / From	1	2	3	4	SUPPLIES
A	15 (30)	18	19	13	50 20
B	20	14 (30)	15	17	30
C	25	12	17	22	70
DEMANDS	30	60	20	40	

If we continue invoking the same jargon, we arrive at;

To / From	1	2	3	4	SUPPLIES
A	15 (30)	18 (20)	19	13	~~50~~ 20
B	20	14 (30)	15	17	~~30~~
C	25	12 (10)	17 (20)	22 (40)	~~70~~ ~~50~~ 10
DEMANDS	30	~~60~~ ~~30~~ 20	20	40	

With the basic feasible total transportation cost

(30x15) + (20x18) + (30x14) + (10x12) +(20x17) +(40x22) =2194.

Comment

If we compare the solution obtained by the two methods, we see that the Minimum cost method is convenient compared to N-W Corner method.

3.4.3 THE VOGEL APPROXIMATION METHOD-VAM

This method is superior over the N-W Corner and the minimum cost methods. The method uses a concept of penalty. The row or column is penalized if allocation is not made to the minimum cost route. It involves computing the row and column penalties. Penalty for any row (or column) is defined to be the difference of the second minimum and the first minimum cost in that row (or column).

i) Let $P_1^r, P_2^r, \ldots P_m^r$ represent row 1,row 2,...,row m penalties
   Let $P_1^c, P_2^c, \ldots, P_n^c$ represent column 1,column 2,... column n penalties.

ii) Determine the maximum penalty by finding maximum
   That is Max $\{P_i^{\,r}, P_j^{\,c}\}$ for all i=1,2,...,m and j=1,2,...,n

iii) Then, if it is in the kth row (s[th] column) determine the least cost in the k[th]
   Row (s[th] column) call this cell x_{ks}

iv) Now, utilize (k,s) route to its full capacity, that is assign $x_{ks} = \min\{s_k, d_s\}$

If $x_{ks} = S_k$ then cross out the k[th] row and change d_s to $d_s - S_k$

If $x_{ks} = d_s$ then cross out the s[th] column and change S_k to $S_k - d_s$

v) Compute new penalties and repeat the process until all rows and columns are crossed out.

Note

a) In case of a tie, choose either row or column with least index.
b) In case of a tie within a row or column choose the cell such that $i + j$ is minimum.

Demonstration of the Vogel's Approximation Method

Consider our previous example;

P_i^r

To / From	1	2	3	4	SUPPLIES
A	**15**	**18**	**19**	**13**	50
B	**20**	**14**	**15**	**17**	30
C	**25**	**12**	**17**	**22**	70
DEMANDS	30	60	20	40	

P_j^c

The first thing is to indicate the penalties which is the difference between the most minimum value and the next to minimum value. Example in the first column, P_1^C=20-15=5

To / From	1	2	3	4	SUPPLIES	P_i^r
A	**15**	**18**	**19**	**13**	50	(2)
B	**20**	**14**	**15**	**17**	30	(1)
C	**25**	**12**	**17**	**22**	70	(5)
DEMANDS	30	60	20	40		
P_j^c	(5)	(2)	(2)	(4)		

We see that, Max $\{P_i^r, P_j^c\}$ =5 and it has appeared in both third row and in the first column. As a rule we choose the row/column with least indices. In this case we choose the first column .In the first column, the least cost is at cell x_{11}, so that the cell $x_{ks} = x_{11}$.

To utilize the route (k, s) we assign the x_{ks} =min $\{s_1, d_1\}$ = min $\{50,30\}$ =30

We cross first column and adjust s_1 to s_1^* =20

If we do that we get the following;

To / From	1	2	3	4	SUPPLIES	P_i^r
A	(30) 15	18	19	13	~~50~~ 20	(2) (5)
B	20	14	15	17	30	(1) (1)
C	25	12	17	22	70	(5)(5)
DEMANDS	~~30~~	60	20	40		
P_j^c	(5)	(2) (2)	(2) (2)	(4) (4)		

We see that Max $\{P_i^r, P_j^c\}$ =5 and we choose a row with least indices again which is the first row.

In the first row, the least cost is at cell x_{14}. To utilize this route we assign the x_{ks} = Min $\{s_1, d_4\}$ = min $\{20,40\}$ =20

Cross 1st row and adjust d_4 to d_4^* =20

To / From	1	2	3	4	SUPPLIES	P_i^r
A	(30) 15	18	19	(20)	~~50~~ ~~20~~	(2) (5)
B	20	14	15	17	30	(1)(1)(1)
C	25	12	17	22	70	(5) (5) (5)
DEMANDS	~~30~~	60	20	~~40~~ 20		
P_j^c	(5)	(2) (2) (2)	(2) (2) (2)	(4) (4) (5)		

Continue in the same fashion, the final tableau will yield the following;

To / From	1	2	3	4	SUPPLIES	P_i^r
A	30 (15)	18	19	13 / 20	~~50~~ 20	(2) (5)
B	20	14	10 (15)	10 (17)	30	(1)(1)(1) (1
C	25	60 (12)	17 / 20	22	70	(5)(5)(5)(5)
DEMANDS	30	60	20	40		
P_j^c	(5)	(2) (2) (2) (2)	(2) (2) (2) (2)	(4) (4) (5)		

The total cost will be
= (30x15) + (60x12) + (10x15) + (10x17) + (20x13) + (20x17)
= 450 + 720 +150 +170 +260 +340
= 2090.

Comparison of Results:

Using North-West Corner we get 2194.
Minimum Cost method gave us 2140.
Vogel's method yielded 2090.

Therefore:
The Vogel is superior of all.

3.5 IMPROVING BASIC FEASIBLE SOLUTION TO OPTIMALITY

In LP simplex method we improve the basic feasible solution by determining the entering and leaving variables. In transportation problem we have the similar procedure but its computations are much simpler.
Pivoting of TP tableau uses the following steps;

1. Determine the variable that will enter the basis
2. Find the loop involving the entering variable and some of the basic variables. It can be shown that there exists only one loop.
3. Count only cells in the loop, label the cells in the loop found in step 2 that are even number of cells away from entering variables as even cells. Also, label those cells in the loop that are an odd number of cells away from entering variables as odd cells.
4. i) Find the odd cell whose variable assumes the smallest value, call this value ϕ .This will leave the basis.

ii) Decrease the value of each odd cell by ϕ and increase the value of each even cell by ϕ

Pivot process is now complete.

iii) Variables not in the loop remain unchanged.

If more than one odd cell in the loop equals ϕ , you may arbitrarily choose one of them to leave the basis .This will result into a degenerate basic feasible solution.

If $\phi=0$ the entering variable will equal 0, and an odd variable that has a current value of zero will leave the basis.

Illustration:

Consider the following example;

2	2	2	1	3
10	8	5	4	7
7	6	6	8	5
4	3	4	4	

If we solve this by Minimum cost method we get.

2	2	2	1 (3)	3
10 (2)	8	5 (4)	4 (1)	7
7 (2)	6 (3)	6	8	5
4	3	4	4	

So that the minimum cost becomes
(2x10) +(2x7) + (3x6) +(4x5) +(1x4) +(3x1)= 20+14+18+20+4+3
= 79.

We can improve this solution to optimality.

s_i

2	2	2	1 (3)	3
10 (2)	8	5 (4)	4 (1)	7
7 (2)	6 (3)	6	8	5
4	3	4	4	

d_j

We introduce the U_i's and V_j's for rows and columns respectively.
In order to determine entering variables we need to find numbers U_i and V_j for all basic variables such that $U_i + V_j = C_{ij}$

Since there are six equations in seven variables,we have to fix one variable to zero. It is common practice to let U_1= 0.

The equations are;

$U_1 + V_4 = 1$
$U_2 + V_1 = 10$
$U_2 + V_3 = 5$
$U_2 + V_4 = 4$
$U_3 + V_1 = 7$
$U_3 + V_2 = 6$
Set $U_1 = 0 \Rightarrow$ $V_4=1; U_2=3, V_1=7; U_3=0; V_2=6; V_3=2$

Then for all non basic variables compute $Y_{ij} = U_i + V_j - C_{ij}$

$Y_{11} = U_1 + V_1 - C_{11} = 0+7-2=5$
$Y_{12} = U_1 + V_2 - C_{12} = 0+6-2=4$
$Y_{13} = U_1 + V_3 - C_{13} = 0+2-2=0$
$Y_{22} = U_2 + V_2 - C_{22} = 3+6-8=1$
$Y_{33} = U_3 + V_3 - C_{33} = 0+2-6=-4$
$Y_{34} = U_3 + V_4 - C_{34} = 0+1-8=-7$

If $y_{ij} = u_i + v_j - c_{ij} \leq 0$ for all non-basic variables ,then the current basic feasible solution is optimal. If this is not the case, select the non basic variable with the most positive yij to enter the basis.

Note

For maximization TP if $y_{ij} \geq 0$ for all non basic variables the current solution is optimal otherwise select the non basic variable with the most negative y_{ij} to enter the basis.

Since our example is a minimization one and that not all $y_{ij} \leq 0$ for the non basic variables then we select x_{11} to enter the basis. The criterion is that we are looking for the most positive value among the y_{ij} .

3.5.1 Loop Determination

The loop that passes at some of the basic variables are cells (1,1) ,(1,4),(2,4) and (2,1).From this we need to determine both even and odd number cells.

Even cells are (1,1) and (2,4) while odd cells are (1,4) and (2,1)

Now, the smallest value in the odd cells is 2 (ie from the basic variables)

$\Rightarrow \phi = 2$ (at x_{21}) so that the basic variable x_{21} leaves the basis.

Tableau.

(2)	2	2	1 (1)	3
10	8	5 (4)	4 (3)	7
7 (2)	(3) 6	6	8	5
4	3	4	4	

Once again we compute the $y_{ij} = u_i + v_j - c_{ij} \leq 0$ for non basic variables

$Y_{12} = U_1 + V_2 - C_{12} = 0+1-2= -1$
$Y_{13} = U_1 + V_3 - C_{13} = 0+2-2 = 0$
$Y_{21} = U_2 + V_1 - C_{21} = 3+2-10= 5$
$Y_{22} = U_2 + V_2 - C_{22} = 3+1-8 = -4$
$Y_{33} = U_3 + V_3 - C_{33} = 5+2-6 = 1$
$Y_{34} = U_3 + V_4 - C_{34} = 5+1-8 = -2$

We note that not all $y_{ij} \leq 0$, the current basic feasible solution is not optimal. The most positive y_{ij} is at x_{33} .Thus x_{33} enters the basis. From the above tableau we need to have a new loop. But after observing the even and odd cells, the even cells are x_{11}, x_{24} and x_{33} ,the odd cells are x_{14} , x_{23} and x_{31} ,in this case $\phi = 1$.

Hence, new tableau upon adding and subtracting ϕ.

(3) 2	(3) 2	2	1	3
10	8	5 (3)	(4) 4	7
(1) 7	6	6 (1)	8	5
4	3	4	4	

If we compute y_{ij} in the same jargon we will find that all $y_{ij} = u_i + v_j - c_{ij} \le 0$.Then we have the following conclusion;

Since all $y_{ij} \le 0$ for all non basic variables current basic feasible solution is optimal;

From	**to**	**amount**	**cost**
O1	D1	3	3x2=6
O2	D3	3	3x5=15
O2	D4	4	4x4=16
O3	D1	1	1x7=7
O3	D2	3	3x6=18
O3	D3	1	1x6=6
TOTAL		**15**	**68**

3.6 DEGENERATE BASIC FEASIBLE SOLUTION

A basic feasible solution V_b (column of basic variables) containing less than $m+n-1$ positive variables is called a degenerate basic solution.

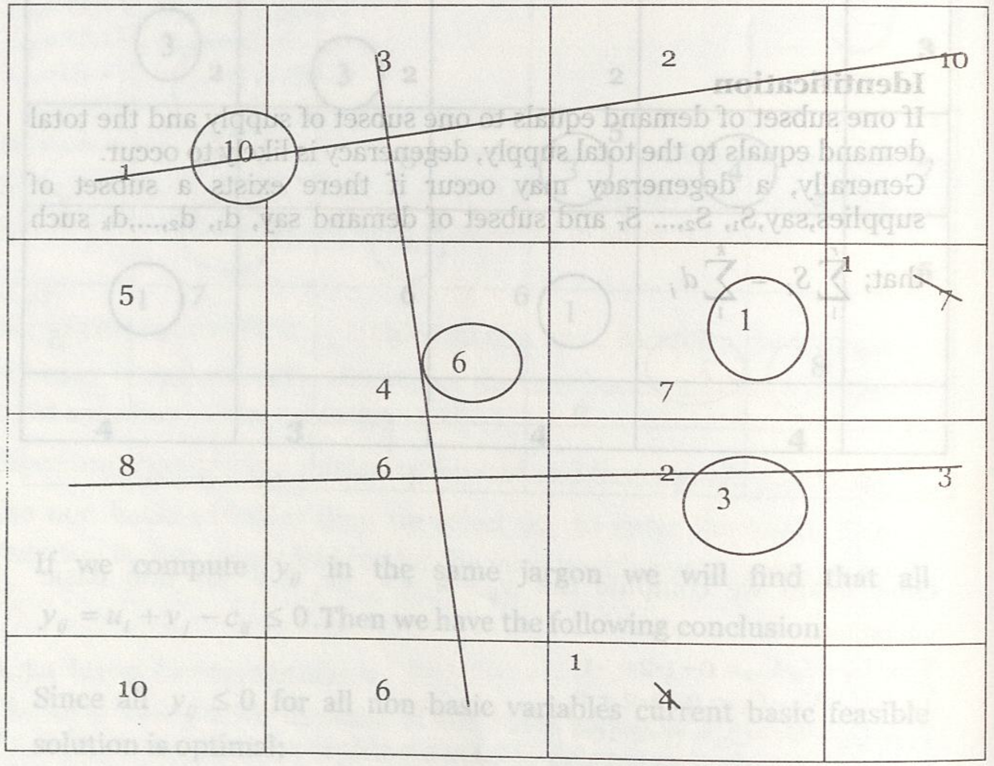
Identification

If one subset of demand equals to one subset of supply and the total demand equals to the total supply, degeneracy is likely to occur.

Generally, a degeneracy may occur if there exists a subset of supplies,say,S_1, S_2,... S_r and subset of demand say, d_1, d_2,...,d_k such that; $\sum_{1}^{r} S_i = \sum_{1}^{k} d_j$

HOW TO DEAL WITH DEGENERACY

Let us utilize the use of minimum cost method to find the basic feasible solution.



x_{11} = min (10, 10) we see that both 1st column and 1st row are satisfied. Let us cross out the 1st row.

The next minimum is at cell x_{33}.

x_{33} =min (3,4) =3. Cross row three and adjust d_3 to d_3^* =1

Next,

x_{22} =min (7,6) =6.Cross 2nd column and adjust s_2 to s_2^*=1.Second column is also satisfied.

Look at the solution above. Clearly, the number of basic variable is five. We know that if $m+n-1 \prec$ the number of basic variables then we have degeneracy (n= number of rows and m= number of columns).So that $3+3-1=5 \succ 4$ .So we have degeneracy and using normal approach we can not proceed.

To get rid of the degeneracy problem we use what is called "*Perturbation Method*" The method introduces a small value denoted by $\mathcal{E}$ as follows;

Either si changed to Si + $\mathcal{E}$ for i= 1...m
dj changed to dj for j=1... n-1
dn changed to dn + m$\mathcal{E}$

Or

dj changed to dj + $\mathcal{E}$ for j= 1...n
si changed to si for i=1... m-1
sm changed to sm + n$\mathcal{E}$

We affect any of these and proceed with our iterations.

In case we have degeneracy during iterations, we assign one of the variables attaining zero a value $\mathcal{E}$ and continue with the iterations.

3.7 ASSIGNMENT PROBLEM

3.7.1 Introduction

Assignment problem is a special case to transportation problem. It is a type of allocation problem in which we associate each of a number of origins with each of the same number of destinations such that each origin is associated with one and only one destination in such a way that the total return resulting from the assignment is optimized.

3.7.2 Representation

Let there be r jobs (J_j; $j = 1,2,...r$) to be assigned to r machines M_i; $i = 1,2,...r$. This implies that we have equal number of jobs and machines and each job has to be assigned in only one particular machine. If job J_j is assigned to machine M_i a return C_{ij} is obtained.

That is;

	J_1	J_2	....	J_r
M_1	C_{11}	C_{12}	....	C_{1r}
M_2	C_{21}	C_{22}	....	C_{2r}
	.	.	.	.
	.	.	.	.
	.	.	.	.
	.	.	.	.
M_r	C_{r1}	C_{r2}	....	C_{rr}

We shall give prohibitive assignment , a very big penalty denoted by M.

Our interest is to find an assignment $x = (x_{ij})$ which optimizes

$$Z = \sum_i^r \sum_j^r c_{ij} x_{ij}$$

$$\text{S/t} \quad \sum_i^r x_{ij} = 1 \quad \text{for i=1..,r}$$

$$\sum_j^r x_{ij} = 1 \quad \text{for j=1...,r}$$

$$x_{ij} = 1 \quad or \quad 0$$

Mathematical representation.

Let $x_{ij} = \begin{cases} 1 & if \quad M_i \quad is\ assigned\ to\ J_j \\ 0 & otherwise \end{cases}$

Optimize

$$Z = \sum_i^r \sum_j^r c_{ij} x_{ij}$$

$$\text{S/t} \quad \sum_i^r x_{ij} = 1 \quad \text{for i=1..,r}$$

$$\sum_j^r x_{ij} = 1 \quad \text{for j=1...,r}$$

$$x_{ij} = (1,0)$$

3.7.3 Unbalanced Assignment Problem

An assignment problem is said to be balanced if the number of resources equals the number of jobs and it is unbalanced otherwise. However, unbalanced AP can be made balanced by adding dummy resource(s) or dummy job(s) with appropriate returns.

Theorem I

If x_{ij} is an optimal solution to AP then also solves AP with new objective function;

$$Z' = \sum_i \sum_j c_{ij}' x_{ij} \quad \text{where}$$

$$C_{ij}' = c_{ij} - p_i - q_j \qquad for \quad i, j = 1,2,...n$$

C_{ij} =returns

P_i = amount associated with rows

Q_j = amount associated with column.

Theorem II

If $C_{ij} \geq 0$ for all i and j, we can find an assignment x_{ij} among the zero's only with total cost $\sum_i \sum_j c_{ij} x_{ij}$ =0 then this assignment is optimal. In here the theorem just illustrates where to make assignments. Also, it is possible to have zero returns (i.e. modified $c_{ij} = c_{ij}' = 0$) that means that, from each row or column a certain amount is to be substracted.

3.7.4 Solution Techniques

For r jobs and r machines, when r is small we can use trial and error approach to obtain the optimal assignment but for large r we use an algorithm to solve it.

3.7.4.1 KUHN'S Method (1950)

This method follows the following steps;

1. From each row subtract it's minimum and in the resulting matrix, from each column substract its minimum.

This refers to $C_{ij}' = c_{ij} - p_i - q_j$ where in here we do consider only a portion $C_{ij} - p_i$ where p_i is the minimum in each row is and $C_{ij} - q_j$ where q_j is the minimum in each column.

Example 2

Let us consider the effective matrix of Assignment problem below. A 4x4 matrix.

A company has four salesmen and four clients to visit. The profit per unit (in US $) is recorded below.

Client	Salesmen A	B	C	D
1	9	12	7	15
2	13	14	15	10
3	8	10	20	6
4	11	15	13	10

Determine the maximum profit by making best assignments.

Solution

We start with row subtraction and we observe that 7,10,6 and 10 are the minimum in rows 1 to 4 respectively. If we do a subtraction we have;

Client	Salesmen A	B	C	D
1	2	5	0	8
2	3	4	5	0
3	2	4	14	0
4	1	5	3	0

Use the above resulting matrix to perform column subtraction, i.e. subtract the minimum to each element in a column.

Client	Salesmen			
	A	B	C	D
1	1	1	0	8
2	2	0	5	0
3	1	0	14	0
4	0	1	3	0

MAKING ASSIGNMENTS

As we said before, for small matrices, the trial and error inspection method can be used in making assignments. In our case we expect to get four assignments.

In case we fail to make assignments from the previous part we then need to add more zeros without making any entry in the matrix negative, that is without violating the rule $c_{ij} \geq 0$.

To affect this, we consider the following by Konig's (1912), Egervange's (1916) and Frobbenicus (1918);

" If the elements of a matrix are divided into two classes by a property ,p, then the minimum number of lines, that contain all the elements with property P,is equal to the maximum number of elements with property P with no "two" on the same line.

In this theorem, is meant that, the maximum number of lines required to cover all zeros equals to the maximum number of assignments possible among zeros.

However, in doing that, there are two problems in which one can encounter;

i) How to draw the lines

ii) Checking for optimality of the current assignment among zeros.

We shall illustrate those procedures using following examples.

Consider the previous example of the four salesmen and four clients.

We had after performing row and column subtraction.

Client	Salesmen			
	A	B	C	D
1	1	1	0	8
2	2	0	5	0
3	1	0	14	0
4	0	1	3	0

In making assignments, we follow the following procedures;

a) Examine rows successively until a row with exactly one unmarked zero is found. Circle this zero as an assignment and mark an (x) all other zeros in the column containing this circled zero, to show that they can't be used to mark other assignments.

b) Examine a column for a single unmarked zero. Circle this zero and mark (x) all other unmarked zeros in its row.

c) Repeat (a) and (b) above until one of the following occurs;

   i) There is no zero left unmarked

   - If the number of circled zeros equal the number of required Assignments, then we have an optimal solution
   - If the number of circled zeros is less than the number of Assignments required, then we have to improve the current solution.

To illustrate the above jargon, we have for part (a);

Client	Salesmen A	B	C	D
1	1	1	(0)	8
2	2	0	5	0
3	1	0	14	0
4	0	1	3	0

For (b) we have;

Client	Salesmen A	B	C	D
1	1	1	(0)	8
2	2	0	5	0
3	1	0	14	0
4	(0)	1	3	X

If we proceed in the same fashion, we have the following final matrix;

Client	Salesman A	B	C	D
1	1	1	(0)	8
2	2	X	5	(0)
3	1	(0)	14	X
4	(0)	1	3	X

Conclusion:

Client 1 will be assigned to salesman C.
Client 2 will be assigned to salesman D.
Client 3 will be assigned to Salesman B.
Client 4 will be assigned to salesman A.

So that we get the best assignment with the total cost equals to US $ 38.

Consider another example, whereby we fail to get the required number of assignment. In this case, we need to create more zeros. This involves, the two problems mentioned earlier, one being how to get the minimum number of lines required to cover all zeros and the other is how to check for optimality of current assignment among zeros.

We are given the following assignment problem with the following effective matrix.

25	18	23	14
38	15	53	23
15	17	41	30
26	28	36	29

Row subtraction gives;

11	4	9	0
23	0	38	8
0	2	26	15
0	2	10	13

Column subtraction gives;

11	4	0	0
23	0	29	8
0	2	17	15
0	2	1	13

Upon observing the third tableau, we fail to get the required number of assignments. In this case we need to create more zeros.

RULES FOR DRAWING LINES

a) Mark all rows for which assignments have not been made
b) Mark all columns not already marked which have zeros in marked rows
c) Mark rows not already marked which have assignments in marked columns
d) Repeat (a),(b) and (c) until the exercise of marking ends
e) Draw lines through all unmarked rows and through all marked columns.

To illustrate the above steps, let us see each step accordingly;

Trial and error solution gave the following results;

11	4	⊗	0
23	⊗	29	8
⊗	2	17	15
0	2	1	3

Where the number of assignments (three) is less than the required which is four assignments.

a)

11	4	⊗	0
23	⊗	29	8
⊗	2	17	15
0	2	1	3

The fourth row is the one in which assignment have not been made, so we mark it is indicated above and we proceed.

b)

11	4	⊗	0
23	⊗	29	8
⊗	2	17	15
0	2	1	3

The first column is the only one which has zero in marked row.

c)

11	4	⊗	0
23	⊗	29	8
⊗	2	17	15
0	2	1	3

d) Draw lines through all unmarked rows and through all marked column. So we have;

11	4	⊗	0
23	⊗	29	8
⊗	2	17	15
0	2	1	3

From the remaining values subtract minimum value among values remaining and then add the minimum value to every element crossed by two lines and leave the rest unaffected then continue with iterations.

If we look in our matrix, the smallest of the remaining value is 1, so we add 1 to the double crossed entries (23 and 11) and we subtract 1 to those uncrossed elements; hence,

12	4	0	X
24	X	29	8
X	1	16	14
0	1	X	2

Note: Those marked X are the required assignments.

Since the required number of assignments is 4 and that we have these in our solution. This implies that the current solution is optimal. The minimum cost associated with this assignment is 14 +15 +15 +36 =80

3.8 EFFECTIVE MATRIX WITH NEGATIVE ELEMENTS

If some entries in the given matrix are negative, then the matrix can be made non-negative, by either of the following two approaches;

3.8.1 ROW/COLUMN APPROACH

In this approach, we select the most negative element of the row (column) say –k where k is a real number, and add k to each element of that row (column).

3.8.2 MATRIX APPROACH

The most negative element of the matrix is selected , say –k where k is a real

Number, and add k to all elements of the matrix.

Note:

For maximization problem, if all elements are positive, then multiply by -1 all the entries and continue with iterations.

Exercise

1. The part authority in charge of habour facilities in Dar es Salaam must assign five Freighters to give deep-water berths for unloading. The cost for five freighters A, B, C, D and E at each of the five berths are shown in the table below:

		Deep – Water Berth				
		1	2	3	4	5
Freighter	A	13	19	9	18	22
	B	10	8	14	7	16
	C	15	18	17	14	28
	D	23	26	19	30	28
	E	16	12	14	13	16

Select the assignment that will minimize the total cost.

2. The soft drinks company is shifting the packed drinks from depots to be closed to depots to be enlarged. The shipping costs are given in the table below:

Depots to be closed	Depots to be enlarged			
	Q	R	S	T
A	3	3	7	9
B	6	5	3	3
C	6	4	8	7
D	5	4	5	4
E	4	3	6	5

Mechanical loaders in five depots to be closed are A: 5, B:7, C:11, D:8 and E:9, additional loaders required at the depots to be expanded are Q:8, R:9, S:11 and T:8 show the initial basic feasible solution by using North West corner method and deduce the minimum transportation cost.

3. Three factories can supply any of six customers with a particular product. The demand for this product from each of the customers 1, 2, 3, 4, 5 and 6 is 40, 35, 25, 20, 60 and 30 tons respectively. Maximum production at factories A, B and C is 60, 70 and 80 tons respectively. The variable production cost per ton is 11.3, 11.0 and 10.8 (USD) at factories A, B and C respectively and the transportation cost per ton from each customer is as shown below:

Factory	Transportation Cost (USD) to Customer					
	1	2	3	4	5	6
A	1.5	1.8	3.1	4.2	2.5	3.0
B	2.2	4.6	3.5	2.4	1.8	4.0
C	3.6	4.8	1.6	4.4	2.8	2.0

(a) Find the cost for each factory (customer pair of producing and transporting one ton from the factory to the customer.

(b) Use minimum cost method to determine the quantity of product to be supplied from each factory to each customer so as to minimize total cost.

4. A company has 4 salesmen and 4 clients to visit. Profits in USD per visit are given below:

Client	Salesmen			
	A	B	C	D
1	9	12	7	15
2	13	14	15	10
3	8	10	20	6
4	11	15	13	10

Determine the best assignments that will maximize the profit.

5. Find the cheapest assignment from the given table below:

Men	Jobs				
	J1	J2	J3	J4	J5
M_1	10	12	13	9	10
M_2	M	11	10	10	14
M_3	9	M	11	11	M
M_4	11	M	M	M	13
M_5	9	10	14	10	M

CHAPTER FOUR

INVENTORY CONTROL

4.0 Introduction

Almost all organizations find it necessary to maintain inventories of goods which are either consumed within the organization or are supplied to users outside the organization. These stocks are costly to maintain. It can be seen that there is an advantage to the level of inventories. I shall be using stock and inventory interchangeably to mean one and the same thing.

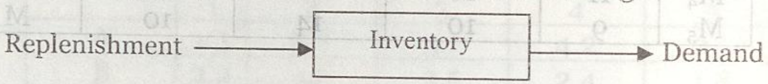


Fig:1. An Inventory System

On the other hand, the primary advantage of having higher inventories is a reduced likelihood of being out of stock (a stock out).The penalty for stock outs may be due to customer dissatisfaction or lost sales. If goods are purchased from a supplier, each additional order will have an additional cost associated with it, such as processing the order, waiting for it, handling the goods when they arrive and so on. In this case there are tradeoffs between ordering larger quantities, thereby maintaining large inventories, and ordering smaller quantities and thereby reducing the inventories. Inventory control attempts to determine policies that balance these opposing strategies.

Definition

Inventory is a list or schedule of materials held- on charge of a person or stock of articles and materials held- on behalf of an organization. The entire office equipment including furniture, fans, lights etc. are the inventory of the office administration. The trucks, the other transports and their associated maintenance facilities and equipment are on inventory charge of transport manager etc. Thus, inventory control means planning, procurement, holding & accounting and distribution of these articles or materials.

This chapter will present the classic economic order quantity models, which represent the most basic and fundamental form of inventory analysis. These models provide a means for determining how much to order and when to place an order so that inventory- related costs are minimized. The underlying assumption of these models is that demand is known with certainty and is constant. In case of unpredictable demand and lead times, probabilistic models were developed in the 1950s to capture their effects. These probabilistic models will not be covered here.

4.1 Types of stock held by business and industry and purpose for which they are held

4.1.1 Raw material stock

By holding stocks of raw material, an organization decouples its primary production sections or processes (e.g. machine shops and press shops) from its raw material manufacturers or stockiest. This allows primary production to be initiated in a shorter period of time than the raw material supplier's delivery time.

4.1.2 Work- in- progress or stock-in-process

The holding of both raw material stocks and stocks of finished goods is generally a planned activity, whereas in-process stocks are likely to exist in any manufacturing organization whether or not they are planned for. The decoupling function provided by this category of inventory is to buffer the demand of a later stage in the production process (e.g. sub-assemblies and final assemblies) from the supply of an earlier stage (e.g. machine shops and press shops); this facility is essential for any production process. Without such decoupling all manufacturing stages would need to be perfectly synchronized- a practical impossibility.

4.1.3 Finished goods

The stocking of finished goods provides a buffer between the customer demand and the manufacturer's supply. In many cases, because the size of orders required by customers are much less than those supplied by the manufacturer, a wholesaler or stockiest can act as intermediary.

The basic function of stock (inventory) is to insulate the production process from changes in the environment. We also note that stock can not only be goods, even human beings can be a stock .Example, stock of money in the bank to be distributed to customers, stock of policemen etc.

The main questions stock controls trying to answer are;

i) How much should be ordered?

ii) When should an order be placed?

In attempting the two questions there are two extremes;

1. A Lot
   - It ensures that we never run out of stock
   - It is an easy way of managing stock
   - It is cheap in managing costs, expensive in stock costs
2. None or Very little
   - Cheap in stock costs, expensive in managing cost
   - Is a difficult way of managing stock

The quantity ordered (how much) is referred to as the Order quantity or Lot size and is denoted by the variable Q. The point in time when an order is to be placed(when) is determined by the inventory level itself. Thus, an order is placed (for Q units) when the inventory falls to a specified level, known as the reoder point.

In dealing with stock models we have basically two main types of costs, holding and ordering costs.

Holding Costs

The cost incurred by keeping inventory, these include:

- Storage costs
- Rent costs
- Labour costs
- Overhead costs, for example,heating,lighting for chickens
- Money tied up (loss of interest)
- Obsolescence costs (if you left with stock at the end of it's product life)
- Depreciation
- Stock deterioration(loss of money if the stock deteriorates whilst held)
- Theft stock

Ordering costs

These are costs associated with placing of an order, these include:

- Clerical or labour costs for processing orders
- Inspection and return of poor quality products
- Transport costs
- Handling costs

Note that a stock out occurs when we have insufficient stock to supply customers. Usually stock outs occur in the order lead time, the time between placing and the arrival of that order.

Given a stock out the order may be lost completely or a customer may choose to backorder that is to be prepared to wait until we have sufficient stock to supply their order.

We shall consider the problem of ordering raw materials stock but the same basic theory can be applied to the problem of:

i) Deciding the finished goods stock and

ii) Deciding the size of a batch in a batch production process.

4.2 THE CLASSICAL ECONOMIC ORDER QUANTITY (EOQ)

We begin with the simplest possible version. Assume a single commodity. Let us consider frequent replenishment by saying that a set up or ordering charge of $\$C$ is incurred each time an order is placed. Let us also consider excessive inventory by saying that each unit will cost $\$B$ to store for 1 unit of time (typically, the time is 1 year, but may be anything provided that other time units are consistent). The "driving force" of the system- the process for which the inventory is held is a steady demand for d units per unit time. All demand must be met without delay.

To keep things simple we assume that replenishment occurs instantaneously whenever an order for replenishment is made (zero lead time).

Suppose that one starts with an inventory of y, the inventory will decrease at the rate d until it reaches zero or an order is placed. If it reaches zero an order must be placed immediately to satisfy all demand without delay. Would there be any reason to order before inventory reaches zero? No, because that would imply that the remaining items were held in inventory (thereby accumulating carrying costs) unnecessarily. If they were needed at all, they could have been ordered in the next shipment and the carrying cost would have been saved. The figure below graphically illustrates the pattern of inventory. We call the order quantity for the first order Q_1. By a repetition of the same argument, we can be sure that inventory will again fall to zero (at the same rate, of course) before the second order, of size Q_2, is placed.

ASSUMPTIONS TO THE MODEL

i) Stock used up at a constant rate(Y units per year)
ii) fixed set up cost C for each order, called the order cost
iii) No lead time/delivery lag between placing an order and arrival of the order
iv) Variable stock holding cost B per unit per year.

Then, we need to decide Q, the amount to order each time, often called the batch or Lot Size which will minimize the total cost. With these assumptions the graph of stock level overtime takes the form shown below;

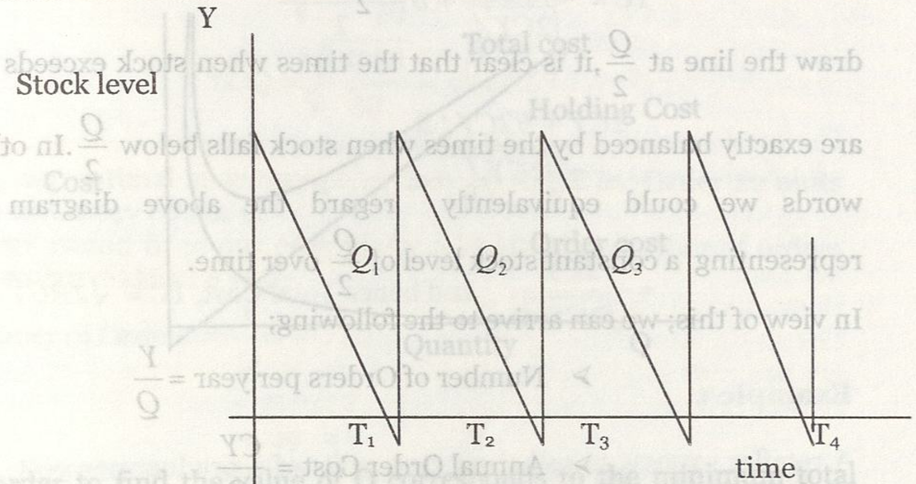


Fig: 2. The Inventory pattern

Now, will Q_2 be larger than, smaller than, or equal to Q_1 ? Without even knowing what determines Q_1 and Q_2, we may argue that they must be equal. Q_1 is determined at time T_1 by taking into account costs and revenues for (possibly) the entire future of the process. But because the future is infinite, and the process is not changing in time, the entire future looks the same at time T_2 as it did at time T_1.

Therefore, however Q_1 is determined, Q_2 will be determined in the same way and will therefore have the same value. By induction, all order sizes are equal, and the inventory level will behave as in Figure 2 above. This "saw tooth" pattern is typical of inventory models.

Since the order size is always the same, we will use Q (without a subscript) to denote the common value and use T to denote the common length of time between orders. Readers are free to use different notations, provided that, they carry the same message at the end.

Consider drawing a horizontal line $\frac{Q}{2}$ in the above diagram. If we draw the line at $\frac{Q}{2}$, it is clear that the times when stock exceeds $\frac{Q}{2}$ are exactly balanced by the times when stock falls below $\frac{Q}{2}$. In other words we could equivalently regard the above diagram as representing a constant stock level of $\frac{Q}{2}$ over time.

In view of this; we can arrive to the following;

- Number of Orders per year = $\frac{Y}{Q}$
- Annual Order Cost = $\frac{CY}{Q}$
- Annual Holding Cost = $\frac{BQ}{2}$

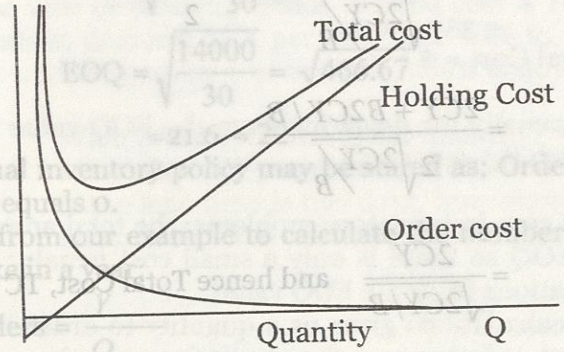
Where Q/2 is the average (constant) inventory level

So that, the total cost = Holding Cost + Ordering Cost

$$TC = \frac{BQ}{2} + \frac{CY}{Q}$$

This is the function that we want to minimize by choosing an appropriate value of Q.

The diagram below illustrates how these two components (annual holding cost and annual order cost) changes as Q, the quantity ordered, changes. As Q increases, holding cost increases but order cost decreases. Hence, the total annual cost curve is as shown below;



In order to find the value of Q corresponds to the minimum total cost, we differentiate the total cost function (TC with respect to Q) and equate to zero.

$$TC = \frac{BQ}{2} + \frac{CY}{Q}$$

$$\frac{dTC}{dQ} = \frac{-CY}{Q^2} + \frac{B}{2} \quad CY/_{Q2} = B/_{2}$$

$\Rightarrow Q^2 = \frac{2CY}{B}$ and thus Q= $\sqrt{2CY/B}$ which is the economic order quantity (EOQ) or Lot Size or batch size.

To obtain the minimum total cost, we subs tract the obtained expression to the Total Cost Equation.

$$TC = \frac{CY}{\sqrt{2CY/B}} + B\frac{\sqrt{2CY/B}}{2}$$

$$= \frac{2CY + B2CY/B}{2\sqrt{2CY/B}}$$

$$= \frac{2CY}{\sqrt{2CY/B}}$$ and hence Total Cost, $TC = \sqrt{2BCY}$

Example 1

A retailer expects to sell about 200 units of a product per year. The storage space taken up in his premises by one unit of this product is costed as $ 20 per year. If the cost associated with ordering is $ 35 per order. What is the EOQ given that interest rates are expected to remain close to 10% per year and the total cost of 1 unit is 100 dollars?

Solution

We are given demand rate, Y = 200 Units per year

The Ordering Cost = $ 35 per order

Holding Cost = $ 20 + Interest x price per unit

= $ 20 +0.1 x 100

= $ 20+10 = $30.

Economic order Quantity, EOQ = $Q= \sqrt{2CY/B}$

$$= \sqrt{\frac{2X35X200}{30}}$$

$$EOQ = \sqrt{\frac{14000}{30}} = \sqrt{466.67}$$

$$=21.6 \approx 22$$

Thus, our optimal inventory policy may be stated as; Order 22 units when inventory equals 0.

We can extend from our example to calculate the number of orders that we can make in a year;

Number of Orders = $\frac{Y}{Q}$

= 200/22

=9.09 ≈ 9

Time between Orders = Reciprocal of the Number of Orders above;

$= \frac{Q}{Y}$ years

= 22/200 years

=0.11 years which is approximately One Month.

The Total minimum cost = $\sqrt{2BCY}$

$= \sqrt{2(35)(30)(200)}$

=648.07

Suppose that for administrative convenience, we ordered 20 and not 22 at each order. What will be our cost penalty for deviating from the EOQ value?

In this case, with Q=20,we recalculate the total annual cost;

$$TC = \frac{CY}{Q} + \frac{BQ}{2}$$

$$= \frac{35(200)}{20} + \frac{30(20)}{2}$$

= 350+300

Total Cost = $ 650

The total penalty for deviating from the EOQ value is $650-$648.2 =$ 1.8 which ii a very small.

This is the case in inventory problems; the total annual cost curve is flat near EOQ so there is only a small cost penalty associated with slight deviations from the EOQ value.

The EOQ calculations gives us a quantity to order but often people are better at ordering on a time basis for example once every month. In that case we can now move from a quantity basis to a time basis.

Consider our previous example where we had EOQ value equals 22.This has an order interval of 22/200=0.11 years, that is we order once every 52 (0.11)=5.72 ≈ 6 weeks. Would you prefer to order once every 6 weeks Or 1 and ½ months? If orders must be made for 1,2,3,4,5 or 12 monthly batches the best order size to use can be determined as follows;

For the 1-month batch the order quantity is the number of components used per month =Y/12=200/12=16.67≈ 17

Since we know the order quantity we can workout the total annual cost of each of the different options and choose the cheapest option. The total annual cost with an order quantity of Q is given by $\frac{CY}{Q} + \frac{BQ}{2}$; if we utilize this we have the following table;

Batch size option	Order Quantity (Q)	Total Annual Cost
Monthly	17	296
2- Monthly	34	531
3-monthly	51	779
4-monthly	68	1030
6-monthly	102	1537
12-monthly	204	3063

The least cost option therefore is to choose a monthly batch.

Example 2

B&M Company is an electronic distributor who has an interest in minimizing the cost of stocking radios. Radios cost $ 10 each, and there is a constant demand of 4 per day. The cost to order more radios is $ 100 and the per day cost to stock radios is 20% of the cost of the radios.

i) What is the optimum order quantity?

ii) How often will you order more radios?

iii) If the lead time is two days, how many radios will be in stock when you order more radios?

Solution

i) EOQ = $\sqrt{2CY/B}$ but in this case $B = i \times p$

$$= \sqrt{\frac{2CY}{ip}}$$

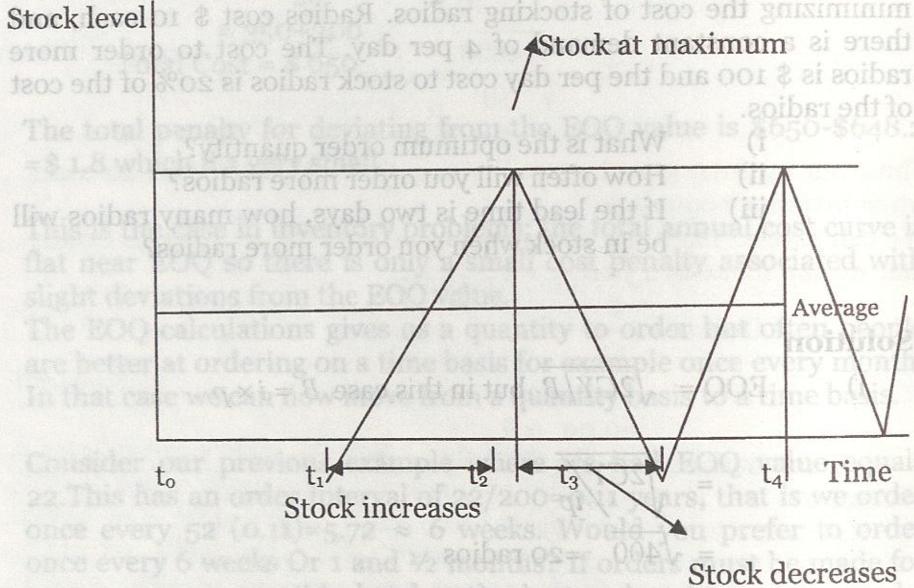
$= \sqrt{400}$ =20 radios

ii) If the demand is 4 radios per day and you order in batches of 20,you will order every Q/Y = 20/4 =5 Days.

iii) If the lead time is L=2 Days, we must replenish our inventory completely in those two days. Since we use 4 units per day, we will order when there are L x Y = 2x4=8 units in stock

4.3 PRODUCTION LOT SIZE MODEL

Sometimes the replenishment of stock occurs gradually, rather than all at once. Such would be the case if the item is produced internally instead of purchased from an outside supplier. Because this result is more aptly described as a lot size than an order quantity, the model is referred to as the production Lot size or PLS model. Consider the diagram below;



Notations:

Let K= Production rate K>Y

C= Set up cost

B= Stock holding cost

Y= Demand rate

Then, Total Set up Cost =C x Number of set ups

$$= C \times \frac{Y}{Q}$$

Stock holding cost =B x Average stock level

$$= B \times \left\{ \frac{(Opening\ stock + Closing\ stock)}{2} \right\}$$

$$= B \times \left\{ \frac{(0 + Closing\ stock)}{2} \right\}$$

Closing Stock = amount produced during t_1 –Amount sold

$$= Q - Sales$$

The time considered here is between to and t1

Closing stock = $Kt_1 - Yt_1 = t_1(K - Y)$

From $Q = Kt_1 \rightarrow t_1 = \frac{Q}{K}$

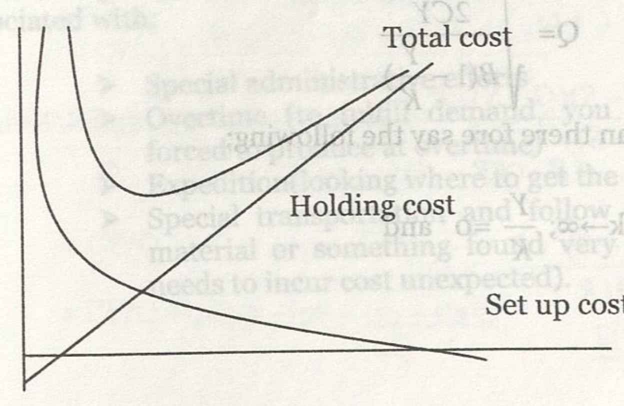
Closing stock = $\frac{Q}{K}(K - Y)$

Thus stock holding cost = B x$\left\{ \frac{Q}{2K}(K - Y) \right\}$

$$= \frac{BQ}{2K}(K - Y)$$

Total Cost = CY/Q + $\frac{BQ}{2K}(K - Y)$

The graph will resemble that we saw earlier when discussed EOQ model.



We need to find Q which minimizes the total cost. To do this we employ calculus and differentiate the total cost function with respect to Q and set it equal to zero.

We know that;

Total Cost = Set up cost + Holding cost

$$TC = CY/Q + \frac{BQ}{2K}(K-Y)$$

$$dTC/dQ = -CY/Q^2 + \frac{B}{2K}(K-Y)$$

$$\frac{CY}{Q^2} = \frac{B}{2K}(K-Y)$$

$$2KCY = BQ^2(K-Y)$$

$$Q^2 = \frac{2KCY}{B(K-Y)}$$

$$Q = \sqrt{\frac{2KCY}{B(K-Y)}} = \sqrt{\frac{2CY}{B(1-\frac{Y}{K})}}$$

Comparing this result with the basic model we had, $Q=\sqrt{\frac{2CY}{B}}$ and in Production Lot size we have;

$$Q=\sqrt{\frac{2CY}{B(1-\frac{Y}{K})}}$$

We can there fore say the following;

When k→∞; $\frac{Y}{K}$ =o and

$$\sqrt{\frac{2CY}{B(1-\frac{Y}{K})}} \rightarrow \sqrt{\frac{2CY}{B}}$$

4.4 ECONOMIC ORDER QUANTITY MODEL WITH SHORTAGES (BACKORDERS ALLOWED)

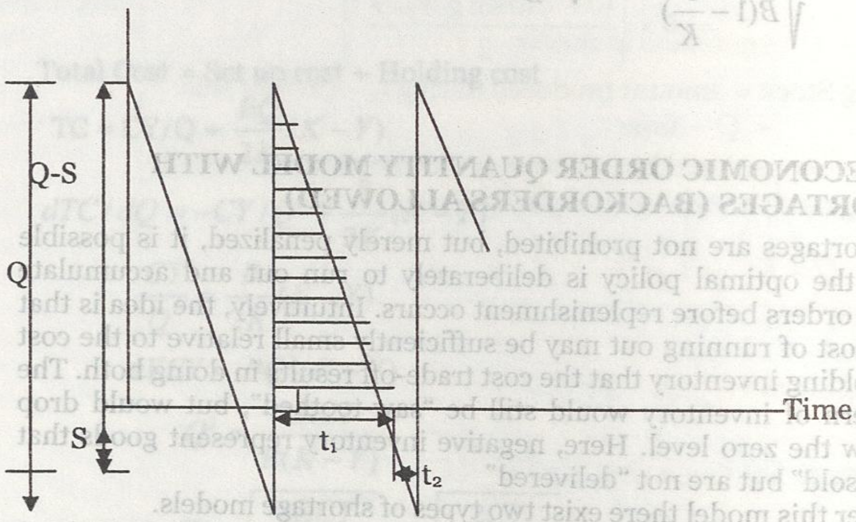
If shortages are not prohibited, but merely penalized, it is possible that the optimal policy is deliberately to run out and accumulate back orders before replenishment occurs. Intuitively, the idea is that the cost of running out may be sufficiently small relative to the cost of holding inventory that the cost trade-off results in doing both. The pattern of inventory would still be "saw-toothed", but would drop below the zero level. Here, negative inventory represent goods that are "sold" but are not "delivered"

Under this model there exist two types of shortage models.

a) Shortage viewed as lost (unfulfilled) demand
b) Shortages are backordered

Irrespective of the type shortage model, there will always be a shortage cost. In the case of lost demand the shortage cost will be the foregone profit (lost profit).In the case of filled demand the shortage costs are associated with;

- Special administrative efforts
- Overtime (to fulfill demand, you may be forced to produce at overtime)
- Expedition(looking where to get the orders)
- Special transportation and follow up (raw material or something found very far, one needs to incur cost unexpected).



Average number of units in stock throughout the year

$$= \frac{\textit{Area of Triangle above time axis}}{\textit{total time}}$$

$$= \frac{1/2(Q-s)(Q-S)/R}{t_1 + t_2} \quad \text{for 1 cycle}$$

$$= \frac{1/2(Q-S)(Q-S)/R}{(Q-S)/R + S/R} \quad \text{where } (Q-S)/R = t_1 \text{ and } S/R = t_2$$

$$= \frac{1/2(Q-S)^2/R}{Q/R}$$

$$= \frac{(Q-S)2}{2Q}$$

Annual holding cost $= \frac{ip(Q-S)^2}{2Q}$

Average number of accumulated (shortage) units throughout

$$= \frac{\textit{Area of triangle below time axis}}{\textit{Total time}}$$

$$= \frac{\frac{1}{2} S * \frac{S}{R}}{\frac{Q-S}{R} + \frac{S}{R}} = \frac{S^2}{2Q}$$

Annual shortage cost $= \frac{Cs\ S^2}{2Q}$ where Cs= shortage cost

We note that, number of orders = Y/Q

Order cost = CY/Q

Therefore;

Total cost = Ordering cost + holding cost+ shortage cost

$$T(Q,S) = C\frac{R}{Q} + \frac{ip(Q-S)^2}{2Q} + \frac{C_s S^2}{2Q}$$

We have;

$$TC(Q,S) = CY/Q + ip(Q-S)^2/2Q + CsS^2/2Q$$

$$= CY/Q + \frac{ip(Q^2 - 2QS + S^2)}{2Q} + \frac{CsS^2}{2Q}$$

$$= \frac{CY}{Q} + \frac{ipQ^2}{2Q} - \frac{ip2Qs}{2Q} + \frac{ipS^2}{2Q} + \frac{CsS^2}{2Q}$$

$$= CY/Q + ipQ/2 - ips + \frac{ipS^2 + CsS^2}{2Q}$$

$$= CY/Q + ipQ/2 - ipS + \frac{(ip + Cs)S^2}{2Q}$$

$$TC(Q,S) = \frac{CY}{Q} + \frac{ipQ}{2} - ipS + \frac{(ip+Cs)S^2}{2Q}$$

We need to find Q and S which will minimize the total cost TC (Q, S)

$$\partial Tc(Q,S)/\partial Q = \frac{-CY}{Q^2} + ip/2 - \frac{(ip+Cs)S^2}{2Q2} = 0 \text{ .... (1)}$$

$$\partial TC(Q,S)/\partial S = -ip + \frac{2(ip+Cs)S}{2Q}$$

$$= -ip + \frac{(ip+Cs)S}{Q} = 0 \text{ .... (2)}$$

From (2) above;

$$-ip + \frac{(ip+Cs)S}{Q} = 0$$ which implies that

$$ip = \frac{(ip+Cs)S}{Q}$$

and $S = \frac{ipQ}{ip+Cs}$ .... (3)

Substitute (3) into (1) we get;

$$\frac{-CY}{Q^2} + \frac{ip}{2} - \frac{(ip+Cs)}{2Q^2} x \frac{(i^2p^2Q^2)}{(ip+Cs)^2} = 0$$

$$\frac{-CY}{Q^2} + \frac{ip}{2} - (ipQ)^2/2Q^2(ip+Cs) = 0$$

$$-CY/Q^2 + ip/2 - \frac{(ip)^2}{2(ip+Cs)} = 0$$

$$-CY/Q^2 + \frac{2(ip+Cs)ip - 2(ip)^2}{4(ip+Cs)} = 0$$

$$\frac{-CY}{Q^2} + \frac{2(ip)^2 + 2C_s ip - 2(ip)^2}{4(ip+Cs)} = 0$$

Upon simplification we have:

$$-CY/Q^2 + \frac{2ipC_s}{4(ip+C_s)} = 0$$

Therefore;

$$\frac{2ipC_s}{4(ip+C_s)} = \frac{CY}{Q^2}$$

So that;

$$Q^2 = \frac{4CY(ip+Cs)}{2ipCs}$$

$$Q = \sqrt{\frac{4CY(ip+Cs)}{2ipCs}}$$ which is also equivalent to;

$$Q = \sqrt{\frac{2CY}{ip}(1+ip/Cs)} = \sqrt{\frac{2CY}{ip}}\sqrt{1+\frac{ip}{Cs}} \text{ .... (4)}$$

Hence S can be written as

$$S = \frac{ipQ}{(ip+Cs)} \text{ but } Q = \sqrt{\frac{2CY}{ip}}\sqrt{1+\frac{ip}{Cs}}$$

$$\text{So } S = \frac{ip}{(ip+Cs)} * \sqrt{\frac{2CY}{ip}}\sqrt{1+\frac{ip}{Cs}}$$

$$S = \sqrt{\frac{(ip)^2}{(ip+Cs)^2} * 2CY/ip * \frac{Cs+ip}{Cs}}$$

Upon simplification we have;

$$S=\sqrt{\frac{2ipCY}{(ip+Cs)Cs}}$$

Divide throughout by ip we get;

$$S=\sqrt{\frac{2CY}{(1+\frac{Cs}{ip})Cs}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

From (4) if Cs (shortage cost) is very big relative to ip (holding cost)

$\rightarrow$ ip/Cs $\rightarrow$ o

It will imply that;

$Q=\sqrt{\frac{2CY}{ip}}$ basic model referred, where $ip = B$

From (5) if Cs is large, then S should be small but when Cs is small then shortage will be very much welcomed.

Example 3

The company has decided to begin manufacturing a part that it has previously being purchased from an outside vendor. The demand is 1000 units per month. The set up Cost per unit is $20, and holding cost is $5 per unit per year.

Once a machine is running, it can produce parts into a rate of 2500 units per month. The company normally operates approximately 300 working days per year. The management would like to know;

8. The production lot size to run
9. Number of production runs
10. The total cost associated with the recommended run size.

Solution

We make all computations on the yearly basis, so;

The demand rate is Y=1000 units per month, this is equivalent to 12000 units per year.

The set up cost, C=$ 20 and the holding cost B=$5

We are also given the production rate, K=2500 units per month =30000 units per year

(1 year=12 months)

So that;

$Q=\sqrt{\frac{2CY}{ip(1-R/K)}}$ note here that ip=B because neither interest nor price is given here.

$$Q=\sqrt{\frac{2\times 20\times 12{,}000}{5\times(1-\frac{12{,}000}{2500\times 12})}} = \underline{400 \text{ units}}.$$

ii) The number of production runs

This can also be referred as the Number of orders

$$=\frac{Y}{Q}=12000/400=30$$

iii) Total Cost;

$$TC= CY/Q+\frac{BQ}{2}(1-\frac{R}{K})$$

$$=\frac{(20)(12{,}000)}{400}+\frac{5(400)}{2}(1-\frac{12{,}000}{30{,}000}) = \underline{\$1200}$$

We can also go further to determine Time between runs ;

$$\frac{Q}{Y}=400/12000=\frac{1}{30} \text{ years}$$

but because we are given 300 working days, then time between runs will be computed as follows;

$$\frac{1}{30}(300) = \underline{10 \text{ days.}}$$

Example 4

Demand for Tents tends to be constant at a rate of 1000 units per month. The per unit carrying cost for shortage and handling is $5 per year. The cost of placing an order is $20.The company is not concerned with shortages, since any unsatisfied demand can be backordered. The management has estimated that shortage cost is approximately $ 0.5 per unit per year.

Compute;

4. The optimal EOQ
5. Maximum Inventory level
6. Total inventory cost

Solution

We are given the following information;

Y= 1000 tents per month=12,000 tents per year

C=$20 and B=$5 per year.

This time;

i) The EOQ = $\sqrt{\frac{2CY}{B}(1+\frac{B}{Cs})}$

$$= \sqrt{\frac{2\times 20\times 12{,}000}{5}(1+\frac{5}{0.5})}$$

$= \underline{1027.60}$

ii) Maximum inventory level is given by simply taking Q-S

We know that S= $\sqrt{\frac{2CY}{Cs}\times\frac{B}{(Cs+B)}} = \sqrt{\frac{2\times 20\times 12000\times 5}{0.5(0.5+5)}}$

S= 934.20

Maximum inventory level= 1027.6-934.20

$= \underline{93.4}$

iii) Total Cost

$$TC = \frac{CY}{Q} + BQ/2 - BS + \frac{(B+Cs)}{2Q}S^2$$

=

$$\frac{20\times 1000\times 12}{1027.6} + \frac{5\times 1027.60}{2} - 5\times 934.20 + \frac{(5+0.5)}{2\times 1027.6}(934.2)^2$$

= 233.55+2569-4671+2335.55

= $\underline{467.1}$

Exercise

1. Kabwaye newspaper publishing concern that must periodically replenish its supply of paperstock . Suppose that the paper comes in large rolls and that the printers use it up at the rate of 32 rolls per week. The cost of replenishment (which includes the cost of bookkeeping, trucking and handling) will be taken to be Tshs. 25,000 plus the cost of the paper. The cost of keeping the paper on hand, including rent for the space occupied, insurance and interest on the capital tied up, will be Tshs. 1000 per roll per week.

   i) What is the Economic Order Quantity (EOQ)

   ii) Calculate the time between orders

   iii) Determine the cost of operating the system

2. a) Prove that for the " Production –run" inventory model, the optimum run size is given by the formula;

$$Q_{optimum} = \sqrt{\frac{2C_sY}{\left(1-\frac{Y}{K}\right)C_h}}$$

Where K= rate of production
Y= demand rate
C_s = Set up Cost
C_h = Inventory holding cost per unit item per unit time.

b) An inventory system that is very closely based on production- run model has the following item values;

K= 4500 articles per year
Y= 1500 articles per year
C_s = 200/=
C_h = 4% of item cost whose price is 40/=

Calculate;

i) Optimum run- size
ii) Cycle length

3. On average Bupalu sells 150,000 units of a product a year which he obtains from a wholesaler. He estimates that the cost of placing an order is Tshs 125,000 with the average inventory storage cost being 20% per year of the cost of a unit (Tshs 12,500). Currently interest rates are 15%.

   a) What would be the optimal order quantity?

   b) Calculate the total inventory cost

   c) Determine the re- order interval

4. A company sells 1,000,000 litres of fuel per month at a constant rate. The fuel is purchased in bulk from a local wholesaler who delivers instantly on demand. A delivery charge of Tshs. 12,000 is made regardless of the order size and the wholesaler price is Tsh. 500 per litre. The cost of capital tied up in stock is estimated at 20% of the average stock value per annum. What are the values for;

   i) Lot size

   ii) Reorder Interval

   iii) Total cost for the Company

CHAPTER FIVE

QUEUING THEORY

5.0 Introduction

Many people have stood in queues at banks, retail shops, bus stops and in other areas where services are being provided and no doubt agree that queuing is a nuisance. In many organizations, the need often arises to provide certain services to a special class of customers or users of the service. These users generally arrive in a random fashion, wait for a service in a queue or waiting line if the service providers are busy, receive the service when they are idle and then leave. These types of environment will be referred to as queuing systems. We shall be looking at some different queuing systems and consider such features as how long it takes on average for a customer to pass through a system from entry to complete service, the average number of customers in the queue at a particular time and the proportion of time that a server is busy.
Queuing theory deals with problems which involve queuing (or waiting);Typical examples includes the telephone switch board in an office, checkout counters in a supermarket, computers waiting for response, public transport(waiting for train or transport),restaurants, printer queuing, hospital appointments, death, etc.

All these constitute one particular and important application of queuing theory. **By analysing these situations to find average waiting-lines, average queue lengths etc, can be obtained for making decisions on such matters as more staff need to be available at particular times of day in order to process customers faster, or whether more queuing space is needed.** However, results of queuing theory have equally, if not more important, applications in industrial situations such as lorries arriving at a factory and needing to queue to be unloaded.

In these types of service environments, the major objective of the decision maker is to provide the service in the best and most efficient manner, thereby generating less congestion and waiting and faster processing of the users.

Why queues?

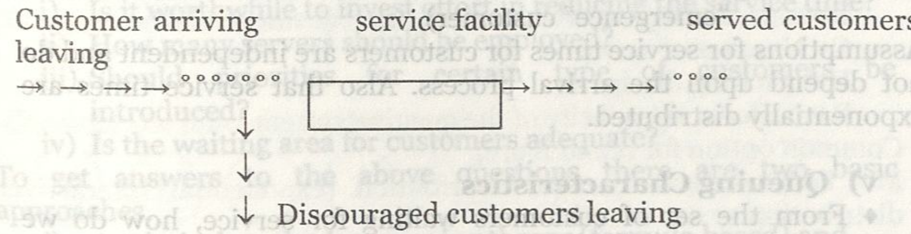
Queues form because resources are limited. In fact it makes economic sense to have queues. For example, how many buses or trains or daladala would be needed if queues were to be avoided or eliminated?

Definition

Queue is a line of customers/ items waiting for a service. It arises in a situation where the arrival rate exceeds the service rate.

5.1 Description of a Queueing System

A queuing system can be described as customers arriving for service, waiting for service if not immediate, and if having wait for service, leaving the service after being served. The schematic diagram of a queuing process is as shown below;



Typically we can talk of this individual sub-system as dealing with customers queuing for service. To analyze this sub-system we need information relating to;

iii) Arrival process

a. How customers arrive example, singly or in groups?

b. How the arrivals are distributed in time (example, what is the probability distribution of time between successive arrivals/the interarrival time distributions).

c. Whether there is a finite population of customers or an infinite number.

The simplest arrival process is one where we have completely regular arrivals (i.e. the same constant time interval between successive arrivals).A Poisson stream of arrivals corresponds to arrivals at random. In a Poisson stream, successive customers arrive after intervals, which independently are exponentially distributed. This stream is important as it is a convenient mathematical model of many real life queuing systems and is distributed by a single arrival rate.

iv) Service Mechanism

a. How long the service will take(the service time distribution)

b. The number of servers available

c. Whether the servers are in series(each server has a separate queue) or in parallel(one queue for all servers)

d. Whether pre-emptions is allowed (a server can stop processing a customer to deal with another "emergence" customer).

Assumptions for service times for customers are independent and do not depend upon the arrival process. Also that service times are exponentially distributed.

v) Queuing Characteristics

♦ From the set of customers waiting for service, how do we choose the one to be served next .That is First-in first out (FIFO), LIFO(Last –in First Out),Mighty is Right, Priority or service in random order. Collectively are known as queuing Discipline.

♦ **Do we have;**

i) balking- customers deciding not to join the queue if it is too long

ii) Reneging- customer leave the queue if they have waited too long for service

iii) Jockeying-customers switch between queues if they think they will get service faster by so doing

Note here that, integral to queuing situations is the idea of uncertainty in, for example, interarrival times and service times. This means that probability and statistics are needed to analyze queuing situations.

In terms of analyzing queuing situations the types of questions in which we are interested are typically concerned with measures of system performance and might include;

iv) How long does a customer expect to wait in the queue before they are served, and how long will they have to wait before the service is complete?

v) What is the average length of the queue?

vi) What is the probability that a customer having to wait longer than a given time interval before they are served.

These are some questions that need to be answered so that management can evaluate alternatives in an attempt to control or improve the situation. Some of the problems that are often investigated in practice are;

i) Is it worthwhile to invest effort in reducing the service time?

ii) How many servers should be employed?

iii) Should priorities for certain type of customers be introduced?

iv) Is the waiting area for customers adequate?

To get answers to the above questions there are two basic approaches.

i) Analytical method or queuing theory (formula based) and

ii) Simulation (computer based)

5.2 PARAMETERS IN QUEUING SYSTEMS AND NOTATIONS

λ (lambda) = mean number of arrivals per time period i.e. The mean arrival rate.
μ (mu) = mean number of customers served per time period. That is, mean service rate.
1/ μ = mean service time for each arrival.
1/ λ = mean time between arrivals.
ρ = traffic intensity/average number being served.
L= Mean number of customers in the system
L_q = Mean number of customers in a queue.
W = Mean time a customer spends in the system
W_q =Mean time in queue.
s =Number of servers
n= State of the queuing system
P_n =Probability that there are n items in the queuing system
P_N =Probability that a customer on arrival is lost.

5.3 Kendal's Notation

Kendal's Notation for a queuing system consists of the five characteristics
A/B/C/D/E
Where
A= represents probability distribution for the arrival process.
B= represents probability distribution for the service process
C= Number of servers
D= Queuing discipline
E= Buffer size/capacity of the queuing system.
Common option for A and B are;
M= For a Poisson arrival distribution (exponential interarrival distribution or exponential
Service time distribution).
D = Deterministic or constant value
G= for general distribution (but with a known mean and variance)
Note;

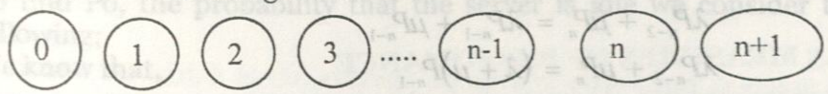
If D and E are not specific then it is assumed that they are infinite.
For example: the M/M/1 queuing system, this is the simplest queuing system, has a Poisson arrival distribution, an exponential service time distribution and a single server.
We also note that in using this notation, it is always assumed that there is just a single queue and customers move from this single queue to the servers.

5.4 SINGLE SERVER SYSTEM

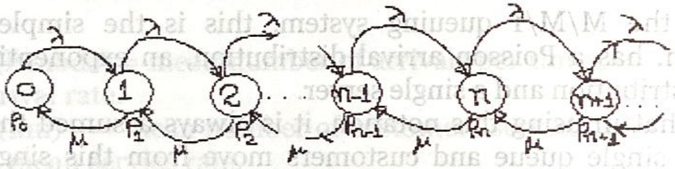
Assume the M/M/1 queuing system.
The central element of the system is a server, which provides some service to items. Items from some population of items arrive at the system to be served. If the server is idle, an item is served immediately. Otherwise, an arriving item joins a waiting line. When the server has completed serving an item, the item departs. If there are items waiting in the queue, one is immediately dispatched to the server. The interest here is on an M/M/1/FIFO/∞ which can be written in short as M/M/1.

In order to derive expressions for L, Lq, W and Wq we first consider the Birth and Death process/Arrival and departure respectively and we assume that the mean number of customers arriving equals those customers departing.
The Rate In= Rate Out diagram helps to derive these parameters.
We also consider a steady state; the rate of arrival does not change.

Consider the following states



In order to have the first customer there should be an arrival and to have no customer there should be a departure. If we effect this we get the following;



At state 0: Rate In = Rate out.

Rate In = μP_1

Rate Out = λP_O

By the property of death and birth process

$$\mu P_1 = \lambda P_O$$

At state 1: Rate In = Rate Out

$$\lambda P_O + \mu P_2 = \lambda P_1 + \mu P_1$$

$$\lambda P_O + \mu P_2 = (\lambda + \mu) P_1$$

At state 2: Rate In = Rate Out

$$\lambda P_1 + \mu P_3 = \lambda P_2 + \mu P_2$$

$$\lambda P_1 + \mu P_3 = (\lambda + \mu) P_2$$

-
-

At state n-1: Rate In = Rate Out

$$\lambda P_{n-2} + \mu P_n = \lambda P_{n-1} + \mu P_{n-1}$$

$$\lambda P_{n-2} + \mu P_n = (\lambda + \mu) P_{n-1}$$

Note:

We assume λ and μ are both known. Then we need to solve the equations above and the criteria is to express the P_1, P_2,... in terms of P_o

If we affect that we get,

From $$\mu P_1 = \lambda P_O$$

$$P_1 = \frac{\lambda}{\mu} P_O \quad \text{.................................} \quad (*)$$

$$\mu P_2 + \lambda P_O = (\lambda + \mu) P_1$$

$$\mu P_2 = \lambda P_1 + \mu P_1 - \lambda P_O$$

But $P_1 = \frac{\lambda}{\mu} P_O$

$$\mu P_2 = \lambda \times \frac{\lambda}{\mu} P_O + \mu \times \frac{\lambda}{\mu} P_O - \lambda P_O$$

$$= \frac{\lambda^2}{\mu} P_O + \lambda P_O - \lambda P_O$$

$$\mu P_2 = \frac{\lambda^2}{\mu} P_O \rightarrow P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_O \quad \text{................} \quad (**)$$

We can do the same for n= 0, 1, 2, 3,...

So that in general;

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_O \qquad n = 1, 2, 3 \ldots$$

To find Po, the probability that the server is idle we consider the following;

We know that,

$$\sum_{n=0}^{\infty} P_n = 1$$

$$\rightarrow \sum_{n=0}^{\infty} \left(\frac{\lambda}{\mu}\right)^n P_o = 1$$

$$P_O = \frac{1}{\sum_{n=0}^{\infty}\left(\frac{\lambda}{\mu}\right)^n}$$

Let

$$\left(\frac{\lambda}{\mu}\right) = \rho$$

$$P_O = \frac{1}{\sum_{n=0}^{\infty}(\rho)^n} \qquad 0 < \rho < 1$$

But;

$$\sum_{n=0}^{\infty}\rho^n = \frac{1}{1-\rho}$$

$$P_0 = \frac{1}{\frac{1}{1-\rho}} = 1-\rho$$

Hence $P_n = \left(\frac{\lambda}{\mu}\right)^n(1-\rho) = \rho^n(1-\rho) \quad for \quad \rho \prec 1$

5.5 MEASURES OF PERFOMANCE

Average number of customers in the system (L).
This can be denoted as $L = E(n)$
Where n = number of customers in the system.
From the definition;

$$E(n) = \sum_{n=0}^{\infty} nP_n$$

$$P_n = \rho^n(1-\rho)$$

$$L = E(n) = \sum_{n=0}^{\infty} n\rho^n(1-\rho) \text{ .... } (*)$$

If we evaluate the series above we set;

$$L = E(n) = (1-\rho)\sum_{n=0}^{\infty} n\rho^n \qquad but \qquad n\rho^n = \rho\frac{d}{d\rho}\times\rho^n$$

$$= (1-\rho)\sum_{n=0}^{\infty}\rho\frac{d}{d\rho}\times\rho^n$$

$$= (1-\rho)\rho\sum_{n=0}^{\infty}\frac{d}{d\rho}\times\rho^n$$

$$= (1-\rho)\rho\frac{d}{d\rho}\sum_{n=0}^{\infty}\rho^n \qquad \text{But} \quad \sum_{n=0}^{\infty}\rho^n = \frac{1}{1-\rho}$$

$$= (1-\rho)\rho\frac{d}{d\rho}\left(\frac{1}{1-\rho}\right) \qquad = (1-\rho)\rho\left(\frac{1}{1-\rho}\right)^2$$

$$L = E(n) = \frac{\rho}{1-\rho}$$

The average number of customers in the system $\quad L = \frac{\rho}{1-\rho}$

To obtain Lq which is the average number of customers queuing or queue length we note that for n customers or units, there will be a queue only if at least one customer /unit is served ,that is (n-1) will be waiting for service. So that

$$L_q = E(n-1) = \sum_{n=0}^{\infty}(n-1)P_n$$

But the easiest way to obtain Lq is by setting;

Lq= L (Average number of customers in the system) – Average being served

$$= L - \rho$$

$$= L - \frac{\lambda}{\mu} \quad \text{But } L = \frac{\rho}{1-\rho}$$

$$= \rho/(1-\rho) - \rho$$

$$= \frac{\rho - \rho(1-\rho)}{1-\rho}$$

$$\rightarrow \frac{\rho^2}{1-\rho} \quad ; \; 0 \prec \rho \prec 1$$

AVERAGE WAITING TIME IN SYSTEM (W):

Suppose:

λ = number of arrivals

W= number of those waiting

Where λ is measured by customer per time.

Then

$L = \lambda W$ is called the Little's Formula and from here we get $W = \frac{L}{\lambda}$

Example 1

At a tool service center, the arrival rate is two per hour and the service potential is three per hour. The hourly wage paid to the attendant at the service centre is $1.50 per hour and the hourly cost of a machinist away from his work is $ 4.00.

a. Calculate the average number of machinists being served or waiting to be served at any given time

b. Calculate the average time a machinist spends waiting for service

c. Calculate the total cost of operating the system for an 8-hour day, if there are two machinists.

Solution

i) We are given that;

Arrival rate, λ = 2 and the service rate μ = 3.We can find traffic intensity which is given by;

$$\rho = \frac{\lambda}{\mu} = \frac{2}{3}$$

Average number of machinists in the system =L

$$L = \frac{\rho}{1-\rho} = \frac{2/3}{1-2/3} = 2$$

ii) Average time spent in the queue is given by Wq

$$W_q = \frac{L_q}{\lambda} = \frac{\rho^2 / 1-\rho}{\lambda}$$

$$= \frac{\lambda}{\mu} \times \frac{1}{\mu - \lambda}$$

$$= \frac{2}{3} \times \frac{1}{1}$$

$$= \frac{2}{3} \text{ hours}$$

<u>Wq = 40 minutes</u>.

iii) Attendant's wage for an 8-hour day is $ (8x 1.50) = $12.On average there are 2

Machinists in the system throughout the day, so 2x8=16 man hours of production are lost. Each of these costs \$ 4.00 so that the total cost of wasted time is 4x16=\$64.So that the total cost of operating this system for an 8-hour day is \$12 +\$ 64 =\$76

Example 2

Materials arrive at the goods-inwards section of a factory at the average rate of five loads per hour. The material is handled by a forklift truck which has an average service rate of seven loads per hour. Management requires to know;

i) The average number of loads at the section waiting to be moved ,when there is a queue
ii) The average length of time an arriving load spends waiting for service
iii) What the average service rate must be in order to reduce the expected time to 20 minutes for a load.

Solution

i) We are given that;

$\lambda = 5/hr$ and $\mu = 7/hr$

Average number in the queue when there is a queue is $\frac{1}{1-\rho}$

$$= \frac{1}{1-\frac{\lambda}{\mu}}$$

$$= \frac{1}{1-\frac{5}{7}}$$

= 7/2=3.5

The average number of loads waiting to be moved when there is a queue =3.5

b) Average time in the queue = $\frac{\rho}{(1-\rho)\mu}$

$$Wq = \frac{\frac{5}{7}}{(1-\frac{5}{7})\times 7}$$

$$= \frac{\frac{5}{7}}{\frac{2}{7}\times 7}$$

$$= \frac{5}{14} \text{ hour}$$

that is, the average time an arriving load spends waiting for service= 21.4 minutes.

c) Let the average service rate needed in order to reduce the waiting time to 20 Minutes be x

Waiting time is denoted by Wq

$$Wq = \frac{\rho}{(1-\rho)\mu} \quad \text{where } \mu = x \text{ in this case}$$

$$= \frac{\frac{\lambda}{\mu}}{(1-\frac{\lambda}{\mu})\times \mu} = \frac{\frac{\lambda}{x}}{(1-\frac{\lambda}{x})\times x}$$

$$= \frac{\frac{5}{x}}{1-\frac{5}{x}} = \frac{1}{3}$$

because 20 minutes = $\frac{1}{3}$ hours

$$\rightarrow \frac{\frac{5}{x}}{\left(\frac{(x-5)}{x}\right)x} = \frac{1}{3}$$

$$\frac{5}{x(x-5)} = \frac{1}{3}$$

$15 = x^2 - 5x$

$x^2 - 5x - 15 = 0$ We solve the quadratic equation above;

x= 7.11, we ignore all negative values in this context.

So we conclude that the service rate must be increased to 7.11 per hour.

Recall that, the Little's formula stated as

$L = \lambda W$

We can use the formula to relate with W as $W = \dfrac{L}{\lambda}$

Average time in queuing (Wq) is given by;

Wq =Average system time- service time

$$= W - \frac{1}{\mu}$$

$$= \frac{L}{\lambda} - \frac{1}{\mu}$$

and sometimes $W_q = \dfrac{L_q}{\lambda}$.

Another important measure of performance is the probability that the server is idle denoted by Po and defined as;

$$P_O = P(server\ is\ idle) = \Pr ob(there\ is\ no\ unit\ in\ the\ system)$$

$$= \Pr ob(n = 0)$$

$$P_O = 1 - P(n \geq 1) = 1 - \rho$$

which follows the fact that;

P (server is utilized) + P (server is idle) =1

P (server is idle) = 1- P (server is utilized)

We note that, a server will be utilized only when there is at least one customer/unit to be processed, So that, in a system of n-units;

$$\Pr ob(server\ is\ idle) = 1 - \Pr ob(n \geq 1)$$

$$P_o = 1 - \Pr ob(n \geq 1)$$

$$= 1 - \rho$$

5.6 FINITE BUFFER SIZE M/M/N

In this case, the L, W, Wq, and Lq changes as compared to infinite buffer.

$$L = E(n) = \sum_{n=0}^{N} n\rho^n P_o$$

$$= P_o \sum_{n=0}^{N} n\rho^n \quad \text{but} \quad n\rho^n = \rho \frac{d}{d\rho} \times \rho^n$$

$$E(n) = P_o \sum_{n=0}^{N} \rho \frac{d}{d\rho} \times \rho^n$$

$$= \rho P_o \sum_{n=0}^{N} \frac{d}{d\rho} \times \rho^n$$

$$= \rho P_o \frac{d}{d\rho} \sum_{n=0}^{N} \rho^n$$

From geometric progression we have;

$$Sn = a\left[\frac{1 - r^n}{1 - r}\right] \text{where r= common ratio}$$

a = first term in the series

$$= \rho P_0 d / d\rho \left[\frac{1 - \rho^{N+1}}{1 - \rho}\right]$$

$$= \rho P_0 \left\{\frac{(1-\rho)d/d\rho(1-\rho^{N+1}) - (1-\rho^{n+1})d/d\rho(1-\rho)}{(1-\rho)2}\right\}$$

$$= \rho P_0 /(1-\rho)^2 \{(1-\rho) - (N+1)\rho^N + (1-\rho^{N+1})\}$$
$$= \rho P_0 /(1-\rho)^2 \{-(N+1)\rho^N (1-\rho) + (1-\rho^{N+1})\}$$
$$= \rho P_0 /(1-\rho)^2 \{-(N\rho^N + \rho^N)(1-\rho) + (1-\rho^{N+1})\}$$
$$= \rho P_0 /(1-\rho)^2 \{-(N\rho^N - N\rho^{N+1} + \rho^N - \rho^{N+1}) + (1-\rho^{N+1})\}$$

Now upon the simplification, we get;

$$E(n) = \rho P_0 /(1-\rho)^2 \{1 + N\rho^{N+1} - \rho^N (N+1)\}$$

We also note that $Po = \frac{1-\rho}{1-\rho^{N+1}}$

So that;

$$E(n) = \frac{\rho \times \frac{1-\rho}{1-\rho^{N+1}}}{(1-\rho)^2} \{1 + N\rho^{N+1} - \rho^N (N+1)\}$$

$$L = \frac{\rho}{(1-\rho)(1-\rho^{N+1})} \{1 + N\rho^{N+1} - \rho^N (N+1)\}$$

This is the expected number of customers in the system.

$$Lq = E(n-1) = \sum_{n=1}^{N} (n-1)\rho^n P_0$$

$$Po \sum_{n=1}^{N} (n-1)\rho^n$$

But (n-1) ρ^n $= \rho^2 d/d\rho(\rho^{n-1})$

$$= P_0 \rho^2 d/d\rho \sum_{n=1}^{N} \rho^{n-1} \quad \text{..................................(*)}$$

Consider $S_n = 1 + \rho + \rho^2 + ... + \rho^{N-1}$

$$\rho S_n = \rho + \rho^2 + ... + \rho^{N-1} + \rho^N$$

Subtracting second equation from first equation we get,

$$S_n - \rho S_n = 1 - \rho^N$$
$$S_n(1-\rho) = 1 - \rho N$$
$$S_n = \frac{(1-\rho^N)}{1-\rho}$$
$$\sum_{n=1}^{N} \rho^{n-1} = \frac{(1-\rho^N)}{1-\rho}$$

Substitute this result into (*) we get

$$= P_o \rho^2 \frac{d}{d\rho}\left[\frac{(1-\rho^N)}{1-\rho}\right]$$

$$= \frac{P_0 \rho^2 \{-N\rho^{N-1}(1-\rho) - (1-\rho^N)(-1)\}}{(1-\rho)^2}$$

$$= \frac{P_0 \rho^2 \{-N\rho^{N-1} + N\rho^N + 1 - \rho^N\}}{(1-\rho)^2}$$

$$Lq = \frac{P_0 \rho^2 \{1 - N\rho^{N-1} + (N-1)\rho^N\}}{(1-\rho)^2}$$

$$= \frac{\rho^2 (1-\rho)}{1-\rho^{N+1}} \left\{\frac{1 - N\rho^{N-1} + (N-1)\rho^N}{(1-\rho)^2}\right\}$$

because $Po = \frac{1-\rho}{1-\rho^{N+1}}$

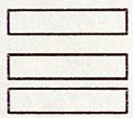
Hence;

$$Lq = \rho^2 \left\{\frac{1 - N\rho^{N-1} + (N-1)\rho^N}{(1-\rho)(1-\rho^{N+1})}\right\}$$

5.7 INTRODUCTION TO MULTI-SERVER QUEUEING SYSTEM

In this situation, a single line is formed in front of s> 1 parallel servers.
Schematic diagram for this can be shown below;

Servers

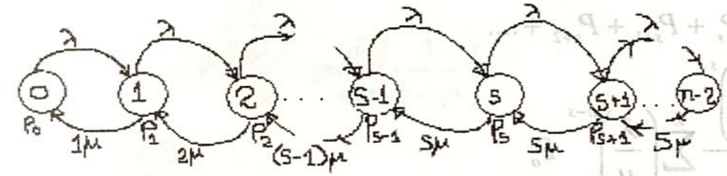
Arrival o o o o o o →  → o o o o o o departure

An arrival goes immediately to an idle server if there is one; otherwise join at the back of the other waiting customers. The assumption is that each server works at the same speed of μ customers per unit of time.

Again, we derive Pn the probability that there are n customers/units in the system. To do this we consider once again a Birth and death process
Let, effective speed be sμ if all servers are busy, μ* the effective speed of service, then

$$\mu^* = \begin{cases} n\mu & \text{if } S \succ n \\ S\mu & \text{if } n \geq S \end{cases}$$

the rate diagram for this is shown below;



At state o : Rate In =Rate Out

$$1\mu P_1 = \lambda P_o \;\rightarrow\; P_1 = \frac{\lambda P_o}{1\mu}$$

At state 1: Rate In =Rate Out

$$\lambda P_o + 2\mu P_2 = \lambda P_1 + 1\mu P_1$$

$$\lambda P_o + 2\mu P_2 = (\lambda + 1\mu) P_1$$

$$P_2 = \frac{\lambda}{2\mu} \times \frac{\lambda}{1\mu} P_o$$

At state s-1: $P_s = \frac{\lambda}{s\mu} \times \frac{\lambda}{(s-1)\mu} \times \frac{\lambda}{(s-2)\mu} \cdots\cdots \frac{\lambda}{2\mu} \times \frac{\lambda}{1\mu} \times P_o$

At state s: $P_{s+1} = \frac{\lambda}{s\mu} \times \frac{\lambda}{(s-1)\mu} \times \frac{\lambda}{(s-2)\mu} \cdots\cdots \frac{\lambda}{1\mu} \times P_o$

At state n-1:

$$P_n = \left[\frac{\lambda}{s\mu} \cdots\cdots\cdots \frac{\lambda}{s\mu}\right] \left[\frac{\lambda}{s\mu} \times \frac{\lambda}{(s-1)\mu} \cdots \frac{\lambda}{1\mu}\right] P_o$$

$$= \left(\frac{\lambda}{s\mu}\right)^{n-s} \left(\frac{\lambda}{\mu}\right)^{s} \frac{1}{s}! P_o$$

In general:

$$P_n = \left(\frac{\lambda}{s\mu}\right)^{n-s} \left(\frac{\lambda}{\mu}\right)^{s} \frac{1}{s}! P_o$$

$$P(n \ge s) = P_s + P_{s+1} + P_{s+2} + \dots$$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!} \sum_{n=s}^{\infty} \left(\frac{\lambda}{\mu}\right)^{n-s} P_o$$

$$P(n \ge s) = \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!} \left(\frac{1}{1-\rho}\right) P_o$$

Average number of customers in queue (Lq)

A queue begins when there are n>s customers, that is n-s customers queuing.

$$L_q = E(n-s)$$

$$= \sum_{n=s}^{\infty} (n-s) P_n = P_o \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!} \sum_{n=s}^{\infty} (n-s) \left(\frac{\lambda}{s\mu}\right)^{n-s}$$

But $\frac{\lambda}{s\mu} = \rho$

$$= P_o \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!} \sum_{n=s}^{\infty} (n-s)(\rho)^{n-s}$$

let $n - s = i$ so that $(n-s)\rho^{n-s} = i\rho^i$ and let also that

$$i\rho^i = \rho \frac{d}{d\rho} \times \rho^i$$

$$= P_o \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!} \sum_{n=s}^{\infty} i\rho^i = P_o \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!} \rho \frac{d}{d\rho} \sum_{i=0}^{\infty} \rho^i$$

$$L_q = P_o \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!} \rho \frac{d}{d\rho} \left(\frac{1}{1-\rho}\right)$$

$$L_q = P_o \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!} \frac{\rho}{(1-\rho)^2}$$

Notations:

- The probability that the whole system is busy = $P(n \ge s)$ this means that all servers are occupied.
- Probability that all servers are idle = $P(n = 0) = 1 - \rho$

In order to obtain those probabilities we rest on the assumption that the sum of all system state probabilities equals to 1, then;

$$\sum_{n=0}^{\infty} P_n = 1$$

We can break the notations into two pieces, that is

$$\sum_{n=0}^{s-1} P_n + \sum_{n=s}^{\infty} P_n = 1$$

$$\sum_{n=0}^{s-1} (\lambda/\mu)^n P_0 / n! + (\lambda/\mu)^s / s! \sum_{n=s}^{\infty} (\lambda / s\mu) P_o = 1$$

so that;

$$P_o = \frac{1}{\sum_{n=0}^{s-1} (\lambda/\mu)^n / n! + \frac{(\lambda/\mu)^s}{s!} \sum_{n=s}^{\infty} (\lambda/s\mu)^{n-s}}$$

Now what is

$$\sum_{n=s}^{\infty} (\lambda / s\mu)^{n-s} \; ; \quad \text{Define } \rho = \frac{\lambda}{s\mu} < 1$$

$$n - s = i; \; n = s = 0$$

$$\sum_{i=0}^{\infty} \rho^i = \frac{1}{1-\rho}$$

$$P_o = \frac{1}{\sum_{n=0}^{s-1} (\lambda/\mu)n \big/ n! + (\lambda/\mu)/s!(1/1-\rho)}$$

From the results above we can now evaluate L.W, Wq

Example 3

Mr. Kingwendu of the Guarantee Bank and Trust Company is the New Assistant Vice President for customer service at the bank. His first assignment is to investigate a new arrangement to shorten the waiting time (queue) for customers being processed by the drive-in teller. Mr. Kingwendu prior standing of the problem has shown that the customers arrive at an average rate of 24 per hour and each drive in teller handles transactions at an average rate of 6 per hour. Because the arrival rate exceeds the service rate the queue will grow so fast, so he consider to have 5 drives in tellers to start with.

Calculate;

1. The probability that the system is empty
2. The probability that the system is busy
3. The average;
   - 3.8 Number of customers queuing
   - 3.9 System time.

Solution

Arrival rate λ =24 per hour
Service rate μ = 6 per hour
Servers s=5

1. The probability that the system is empty

$$Po = \frac{1}{\sum_{n=0}^{s-1} (\lambda/\mu)^n \big/ n! + (\lambda/\mu)^s / s!(1/1-\rho)}$$

We also note that $\frac{\lambda}{s\mu} = \rho = \frac{24}{30} = \frac{4}{5} = 0.8$

$$\frac{\lambda}{\mu} = \frac{24}{6} = 4$$

Upon the substitution of the above results P_o= 0.013.

2. The probability that the system is busy= $P(n \geq s)$

$$= P(n \geq 5)$$

$$P(n \geq 5) = \frac{\left(\frac{\lambda}{\mu}\right)^s}{s!} \left(\frac{1}{1-\rho}\right) P_o$$

$$\frac{\lambda}{\mu} = 4; \; \rho = 0.8 \quad and \quad P_o = 0.0130$$

So that $P(n \geq 5) = 0.555$

3. Requires to compute Lq first;

3. $$L_q = \frac{4^5}{5!} \times \frac{0.8}{0.2^5} (0.0130) = 2.219$$

3. $$W = W_q + \frac{1}{\mu}$$

$$= \frac{L_q}{\lambda} + \frac{1}{\mu}$$

$$= \frac{2.201}{24} + \frac{1}{6}$$

$=0.092 + 0.167$

$= 0.259$ hours

So W= 15.54 minutes

5.8 M/G/1 Model

In this model we have;

M= Poisson arrival process with intensity λ

G= General Service time distribution with mean $\frac{1}{\mu}$

1= Single server

An infinite Buffer is also assumed in this model.

In this model, it is better to know the average service time E(s) and the variance of service time V(s). To make use of these, one particular result which is exact and neat is obtained using Pollaczek-Khintchine (P-K) formula. The result is valid for steady state condition $\left(\frac{\lambda}{\mu} \prec 1\right)$

Mean number in system (L):

$$L = \lambda E(s) + \frac{\lambda^2[(E(s))^2 + V(s)}{2(1 - \lambda/\mu)}$$

Where E(s) = mean service time

V(s) =Variance of service time

The knowledge of mean and variance of service time is crucial to this model.

5.9 VARIATIONS OF M/G/1 MODEL

Let the number of servers be infinite (self service system) that is M/G/∞

Mean number of servers in the system = λ E (s)

Because there will be no customers queuing. We do ignore the average queue length portion in the formula.

Let the service time be deterministic, then because of the constant behavior, the variance will take the value zero, that is V(s) =0.

If that is the case;

$$L = \lambda E(s) + \frac{\lambda^2[(E(s))^2}{2(1 - \lambda/\mu)}$$

5.10 JACKSON'S NETWORK

Jackson's Theorem.

1. The number of customers N_i in different nodes i=1, 2, 3,..., M are independent.
2. Queue i behaves as if the arrival stream N_i were poissonian

State Vector.

The network state is determined by the vector N such that N= $(N_1, N_2, ..., N_M)$

It's possible values are denoted by n = $(n_1, n_2, ..., n_m)$

Let $P(\underline{n}) = P(\underline{N} = \underline{n})$

Define $P(\underline{n}) = 0$ if $n_i \prec 0$

Since the queues are independent and that the flow (arrival) stream is random, then; Jackson's theorem in mathematical term can be written as

$$P(\underline{n}) = P_1(n_1) \times P_2(n_2) \times .... \times P_n(n_m)$$

$$= \prod_{i=1}^{n} P_i(n_i)$$

which is known as a *PRODUCT FORM SOLUTION OR TRADITIONAL SOLUTION.*

For instance if the queues are M/M/1

$$P(n) = \prod_{i=1}^{M} \rho_i^{n_i}(1-\rho_i)$$

If $\rho^i = \rho$ traffic intensity is the same, then

$$P(n) = \prod_{i=1}^{M} \rho^{n_i}(1-\rho) \qquad \rho \prec 1$$

Example 4

Consider M/M/1 service system consisting of 3 stages where λ = 0.4, μ = 2

Find;

- $P(n_1 = 3, n_2 = 4, n_3 = 2)$
- Mean number of customers queuing in all 3 stages

Solution

In the above problem, we notice the following system

λ=0.4 S₁=1 S₂=1 S₃=1

→ 000000→ [ ] 00000 →[ ] 000→[ ]0000→

μ₁=2 μ₂=2 μ₃=2

$$\rho = \rho_1 = \rho_2 = \rho_3 = \frac{\lambda}{\mu} = \frac{0.4}{2} = 0.2$$

$$P(n_i) = \rho^{n_i}(1-\rho) \qquad i = 1,2,3.$$

i) $P(n_1 = 3, n_2 = 4, n_3 = 2) = \prod_{i=1}^{3} \rho^{n_i}(1-\rho)$

$$= \prod_{i=1}^{3} (0.2)^{n_i}(0.8) \qquad i = 1,2,3.$$

$$= (0.2)^3(0.8)\times(0.2)^4(0.8)\times(0.2)^2(0.8)$$

$$P(n_i) = (0.2)^9(0.8)^3$$

$$P(n_i) = 2.62144\times10^{-7}$$

- The average number of customers queuing.
  This is the mean number of customers queuing is denoted by Lq:

$$Lq = \sum_{i=1}^{3} Lq_i$$

$$= L_{q_1} + L_{q_2} + L_{q_3}$$

but $Lq = \frac{\rho^2}{1-\rho} = \frac{(0.2)^2}{0.8} = \frac{0.04}{0.8}$

$=3(0.05)$

$$L_q = 0.15$$

Exercise

1. A department store has a single server cashier. During the rush hours, customers arrive at a rate of 20 customers per hour. The average number of customers that can be attended by the cashier is 24 per hour. Assume that the conditions for use of the single channel queuing model apply;
   i) What is the probability that the cashier is idle?
   ii) What is the average number of customers in the queuing system?
   iii) What is the average number of customers in the queue?
   iv) What is the average time a customer spends in the system?
   v) What is the average time a customer spends in the queue waiting for the service?

2 (a) Kendal's notation for queuing system consists of five characteristics, which are denoted by five- letter- symbol A/B/C/D/E. What does each of letter-symbols stand for?

(b) Use Kendal notation to symbolize each of the following situations;

i) Bottles coming off an assembly line at a constant rate to be inspected. The inspection time is random in length and there are four inspectors.

ii) Students randomly arriving to pre- register for semester I of an academic year. The registration time is random in length and there is one advisor available for registering.

3. In a study of a local hamburger emporium known as Mama Lishe, students in operations customers all stand in a single line to place and receive their orders, the time to serve each customer is random in length. The students then collected data on arrivals and service times. Arrivals were observed for

a 1- hour period and the number of arrivals was noted during each 10- minute period during an hour. The results of this check are as follows;

Interval (minutes)	Arrivals
0 – 10	14
10 – 20	5
20 – 30	10
30 – 40	8
40 – 50	12
50 - 60	7

For a random sampling of the above arrivals, the service times (in seconds) were as follows;

25	20	15	20	45	25
13	28	55	52		
35	25	32	65	25	45
25	10	30	38		
42	55	13	30	45	85
70	45	10	58		
30	50	15			

Using the assumption that these arrivals and service times do indeed fit the Poisson and Negative Exponential distributions;

a) Calculate the average arrival rate and service rate.
b) Determine the average number of customers queuing.
c) Determine the average system time.

4. At a certain airport it takes exactly 5 minutes to land a plane once it is given the signal to land. Although incoming planes have scheduled arrivals times, the wide variability in arrival times produces an effect, which makes the incoming planes appear to arrive in a Poisson fashion at an average rate of six per hour. This produces occasional stack- ups at the airport, which can be dangerous and costly.

i) What is the expected number of planes that are either waiting to land or landing?

ii) What is the expected time from arriving at the airport to completing the landing operations?

5. A launderette has three washing machines. Customers arrive at random at a rate of 4 per hour to use the washing machines, and they queue if all the washing machines are busy. Each customer just uses one washing machine and on average it takes the machine 30 minutes to wash ones customer's clothes.

i) Assume negative exponential distributions of washing times, what is the probability that the three washing machines are all busy at the same time?

ii) How many customers on the average will be waiting for an idle machine?

CHAPTER SIX

NETWORK ANALYSIS

CPM & PERT

6.0 Introduction

Network analysis is one of the most popular techniques used for planning, scheduling, monitoring and coordinating large and complex projects comprising a number of activities. It involves the development of a network to indicate logical sequence of work content elements of a complex situation. It involves three basic steps:

Defining the job to be done

Intergrating the elements of the job in a logical time sequence

Controlling the progress of the project

Network analysis is concerned with minimizing some measure of performance of the system such as the total completion time for the project, overall cost and so on.

The network techniques that are used for project analysis are CPM and PERT. CPM stands for Critical Path Method and PERT stands for Project Evaluation and Review Technique. These two techniques are basically identical, except that PERT is a probabilistic technique whereas CPM is a deterministic (non probabilistic) technique. The fact that they have already been so frequently and widely applied attests to their value as management science techniques.

PERT and CPM have been used to plan, schedule and control a wide variety of projects such as;

Research and development of a new products and processes

Construction of plants, buildings, highways

Maintenance of large and complex equipment

Design and installation of new systems

In projects like the mentioned above, projects managers must schedule and coordinate the various jobs or activities so that the entire project is completed on time. A complicating factor in carrying out this task is the interdependency of the activities; for example, some activities depend upon the completion of other activities before they can be started. For example in building a classroom, say, an activity "erect a wall" must finish before an activity "put a roof" begins. When we realize that projects can have as many as several thousand specific activities, we see why project managers look for procedures that will help them answer questions such as;

What is the total time to complete the project?

What are the scheduled start and finish dates for each specific activity?

Which activities are "critical" and must be completed exactly as scheduled in order to keep the project on schedule?

How long can "non critical" activities be delayed before they cause a delay in the total project?

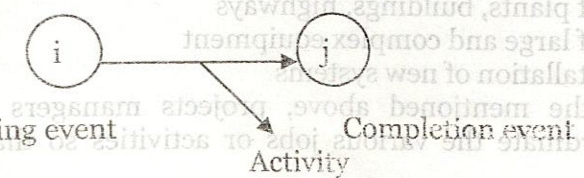
As you will see, PERT and CPM can be used to help answer the above questions.

6.1 CPM/ PERT Network Components

CPM/ PERT networks consists of two major components as discussed below:

Activity

An activity represents some action and as such it is a time consuming part of the project. It may be an operation, transportation or inspection. It consumes both time and resources. It should be noted that an activity can be represented by an arrow or node. Each and every activity has a point of time where it begins and a point where it ends. All the discussion in this chapter will be on " activity on arc" as shown below;



Event

An event represents the start (beginning) or completion (end) of some activity and as such it consumes no time. It has no time duration and does not consume any resources. It is also known as a node. An event is represented by a circle. An event is not complete until all the activities flowing into it are completed. Activities are thus identified by the numbers of their starting (tail or initial) event and ending (terminal or head) event. An arrow (i, j) extended between two events, the tail event i represents the start of the activity and the head event j, represent the completion of the activity as shown in above figure.

6.2 Classification of activities

Predecessor activity

An activity which must be completed before one or more other activities start is known as predecessor activity.

Successor activity

An activity which started immediately after one or more of other activities are completed is known as successor activity.

Dummy activity

Activity which does not consume either any resource and time is known as dummy activity. A dummy activity is depicted by dotted line in the network diagram.

A dummy activity in the network is added only to represent the given precedence relationships among activities of the project and is needed when ;

(a) two or more parallel activities in a project have same head and tail events or (b) two or more activities have same (but not all) of their immediate predecessor activities in common.

6.3 CRITICAL PATH METHOD (CPM)

CPM was developed to help scheduling of routine plant overhaul and maintenance, building or a pilot model plant, etc. It is based on the assumption that the time which each activity in the project will take is precise and unknown. The relation between the amount of resources employed and the time needed to complete the project is also assumed to be known.

Definition

Critical Path in a network diagram is the longest continuous chain of activities (i.e a path along which it takes the longest duration) through the network starting from first to the last event and is shown by thick line or normally double lines normally. All activities lying in this critical path are called critical activities as any delay in their execution will lead to a delay in the completion of the entire project.

6.3.1 Methodology of CPM

The iterative procedure of determining the critical path involves the following steps:

- Break down the project into various activities systematically .
- Arrange all activities in logical sequence and label them. Construct the arrow diagram
- Number all the events and activities. Find the time for each activity considering it to be deterministic. Indicate the activity times on the arrow diagram.
- Calculate the earliest start times and earliest finish time, latest start time and latest finish time.
- Determine total float for each activity on the basis of difference between the earliest time and latest time.
- Identify critical activities and you may connect them with double line arrow. This gives the Critical Path.
- Calculate the total duration of the project

6.4 Critical Path Analysis

The objective of critical path analysis is to estimate the total project duration and to assign starting and finishing times to all activities involved in the project. This helps in checking actual progress against the scheduled duration of the project.

6.4.1 Time Estimates in Network Analysis

Time analysis of the network can be done for planning various activities of the project, once the network model of a project is constructed. For doing so, an estimate of time to complete an activity is required. Activity time is the forecast of the time an activity is expected to take from its starting time to its completion, under normal conditions. For example, an activity might be expected to take 9 weeks using one person working one shift per day and 6 working days per week. Naturally, if we allocate two persons for the same job, time requirement will change. Estimation of activity time is rather difficult. Here we assume that for each activity, time estimates are available in time units and these estimates will be put on top/ bottom of each arrow in the diagram.

The basic objective of the time analysis is to get a planned schedule of the project. The plan should include:

- Total completion time for the project
- Earliest time when each activity can begin
- Latest time when each activity can be started, without delaying the total project
- Float for each activity, i.e. amount of time by which the completion of an activity can be delayed without delaying the total project completion
- Identification of critical activities and critical path

Let us now define some notations;

(i, j) = Activity (i, j) with tail end number i and head end event number j

$T_e or E_i$ = Earliest event occurrence time of event i

$t_L or L_j$ = Latest allowable event occurrence time of event j .

t_{ij} = Time estimate of activity (i, j)

ES_{ij} = Earliest starting time of activity (i, j)

EF_{ij} = Earliest finish time of activity (i, j)

LS_{ij} = Latest starting time of activity (i, j)

LF_{ij} = Latest finish time of activity (i,j)

The basic scheduling computations can be grouped under the following three heads;

a. Forward Pass (Earliest event time)

Based on a fixed occurrence time of the initial network event, the forward pass computation yields the earliest start and earliest finish times for each activity and indirectly the earliest expected occurrence time for each event.

i. The computations begin from the 'start' node and move to the 'end' node. To accomplish this, the forward pass computations start with an assumed earliest occurrence time of zero for the initial project event.

ii. i) Earliest starting time for activity (i, j) is the earliest event time of the tail end event, i.e.; $ES_{ij} = E_i$.

ii) Earliest finish time of activity (i, j) is the earliest starting time plus the activity time; i.e.: $ES_{ij} + t_{ij}$

iii) Earliest event time for event j is the maximum of the earliest finish times of all activities ending into that event. Thus

$$Ej = Maximum\left(ES_{ij} + t_{ij}\right) = Max\left(E_i + t_{ij}\right)$$

$= Maximum$ {EF for all immediate predecessors of (i, j)}

The computed values are put into the lower left portion of each event.

b. Backward Pass. (Latest allowable time).

The latest event times (L) specifies the time by which all activities entering into that event must be completed, without delaying the total project. These are computed by reversing the method of calculation used for earliest event times.

Assume $L = E$ for the ending event.

Latest finish time for activity (i, j) equal to the latest event time of Event j i.e., $LF_{ij} = L_j$.

Latest starting time of activity (i, j) is the latest completion time of (i, j) minus the activity time, i.e.; $LS_{ij} = LF_{ij} - t_{ij}$

Latest event time for event i is the minimum of the latest start time of all activities originating from that event. Thus; $L_i = Maximum\left(LS_{ij}\right)$

$= Min\left(LF_{ij} - t_{ij}\right) = Min\left(L_j - t_{ij}\right)$

$= Min$ {LS for all immediate successors of (i, j)}.

The Computed values are put into the lower right portion of each event.

Advantages and disadvantages of CPM

Advantages:

CPM highlights the critical activities on which management should focus attention to reduce project completion time

It helps management in diverting resources from non- critical to critical activities, i.e it facilitates optimum allocation of resources

It provides a technique of planning and scheduling a project. Scheduling helps to determine completion date and to evaluate progress towards the completion of the project.

It gives complete information about the significance, size, duration and performance of an activity.

CPM does not incorporate statistical analysis in determining time estimate. At least it becomes convenient for the non-mathematicians.

Disadvantages:

CPM operates on the assumption that there is a precise known time that each activity in the project will take. But this may not be true in real life situations.

Each time changes are introduced into the network the entire evaluation of the project has to be repeated and a new critical path has to be determined.

CPM is not suitable for a situation which does not have definite start and definite finish.

6.6 PROGRAMME EVALUATION AND REVIEW TECHNIQUE (PERT)

The protagonists of PERT argue that it is impossible to be certain that a job will take exactly three days or six weeks or whatever duration is assumed in a critical path method. When duration of activities is uncertain, we use the PERT Method. Under this method the activity completion time is specified using three considerable times.

The Optimistic time.

This is the time duration that is the shortest possible for completing the activity, and we shall denote this by lower case (a)

The most likely time

This is the time duration that has the highest probability of occurrence. It shall be denoted by (m)

The Pessimistic estimate

The time duration that is the longest possible for completing the activity. It shall be denoted by lower case (b)

A more accurate model of the duration of an activity is to describe the duration by a probability distribution. One probability distribution that has been used extensively to model activity durations is the Beta distribution. PERT assumes that the random variable T_{ij} which represent the duration of the activity j linked to activity i, that is you can't complete j before i is complete, follows a Beta distribution and if so, it can be shown that.

$$E(T_{ij}) = \frac{a+4m+b}{6} \quad \text{and} \quad Var(T_{ij}) = \left(\frac{b-a}{6}\right)^2$$

The assumption on the activity duration is that, they are independent.

Independence here implies that, time spent for the completion of the predecessor activity has nothing to do with the time of completion of the successor activity.

Then, for any path in the project network, the mean and variance of the time required to complete the activities on the path are given by;

$\sum E(T_{ij})$ =Expected duration of activities on any path. The summation here is done all over the path.

$\sum Var(T_{ij})$ =Variance of duration of activities on any path. Also the summation is done through the path.

If we let T be a random variable denoting the total duration of the activities on a critical path found by CPM.PERT assumes that the critical path found by CPM contains enough activities to allow us to invoke the Central limit theorem and conclude that;

$T = \sum T_{ij}$ is normally distributed.

From these result we can now judge as what is the chance that the task will be completed within a certain interval of days/time.

6.6.1 Advantages and disadvantages of PERT

Advantages:

It focuses attention on critical or bottleneck elements of the project so that a manager may either allocate more resources to them to keep a careful watch on them as the project progress. It permits control by exception and better management of resources.

PERT incorporates the statistical analysis in determining time estimates and enables determination of the probabilities concerning the time by which each activity as well as the entire project would be completed. As such it may be considered an advancement over CPM.

PERT makes possible the pressure for action at the right spot and level in the organization at the right time. It suggests areas for increasing efficiency and reducing cost.

Disadvantage:

It is not practicable for routine planning of recurring activities. It is useful in complex projects consisting of numerous activities which are independent of each other and whose completion times are uncertain.

Time estimates to perform activities constitute a major limitation of PERT.

In PERT, probabilities are calculated on the assumption that a large number of independent activities operate on critical path and as such the distribution of the total time is normal. This assumption may not be true in real life situations.

It is not that, the above are the only advantages and disadvantages of the CPM and PERT techniques, a reader may think of others whenever found upon working with the two techniques.

Distinction between PERT and CPM

In fact both PERT and CPM are managerial techniques for planning and control of large complex projects. Both are techniques of network analysis wherein a network is prepared to analyse interrelationships between different activities of a project. However, there are several differences between the two techniques:-

CPM is used for repetitive jobs like planning the construction of a house. On the other hand, PERT is used for non-repetitive jobs like planning and assembly of the space platform.

CPM is a deterministic model with well known activity (single) times based upon past experience. It, therefore, does not deal with uncertainty in project duration. PERT is a probabilistic model with uncertainty in activity duration, with multiple time estimates made to calculate the probability of completing the project within scheduled time.

PERT incorporates statistical analysis and thereby enables the determination of probabilities concerning the time by which each activity and the entire project would be completed. On the other hand, CPM does not incorporate statistical analysis in determining time estimates because time is precious and known.

PERT is applied mainly for planning and scheduling research programmes. On the other hand, CPM is employed in construction and business problems.

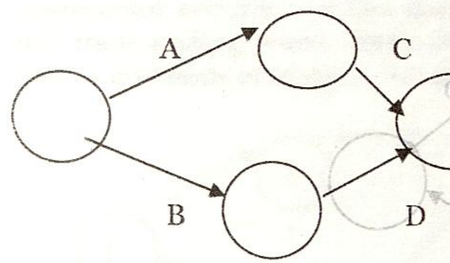
6.7 Worked Examples

Example 1

Given the following information, develop a network

Activity	Immediate Predecessor
A	-
B	-
C	A
D	B

Solution

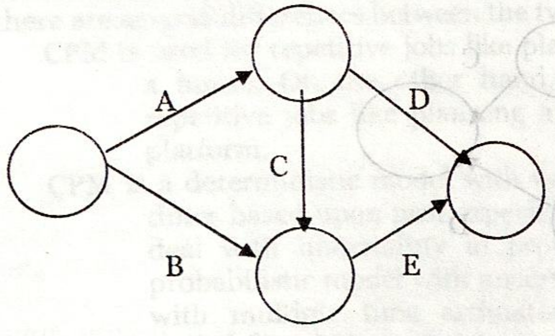


Example 2

Given the following information, develop a network

Activity	Immediate Predecessor
A	-
B	-
C	A
D	A
E	C,B

Solution



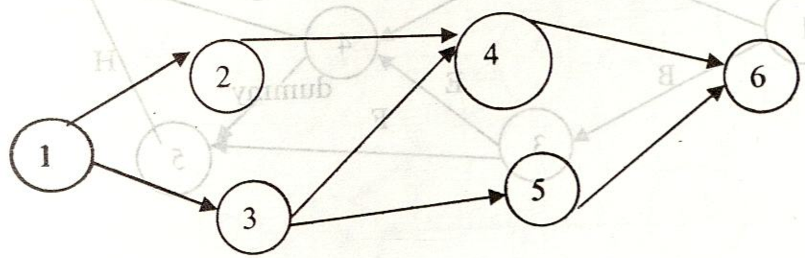
Example 3

Given the following table, develop a network

Beginning Event	End Event	Activity
1	2	1-2
1	3	1-3
2	4	2-4
3	4	3-4
3	5	3-5
4	6	4-6
5	6	5-6

Solution

Here, instead of using a letter to signify activities and their predecessor activities we can specify activities by their starting event and their ending event. Beginning with the activity that starts at event 1 and ends at event 2, we can construct the adjoining network.



Example 4

Construct an activity on arc (network) diagram for the following dependency table of a particular project.

Activity	Prescribed by
A	-
B	-
C	A
D	A
E	B
F	B
G	C &E
H	C,E &F

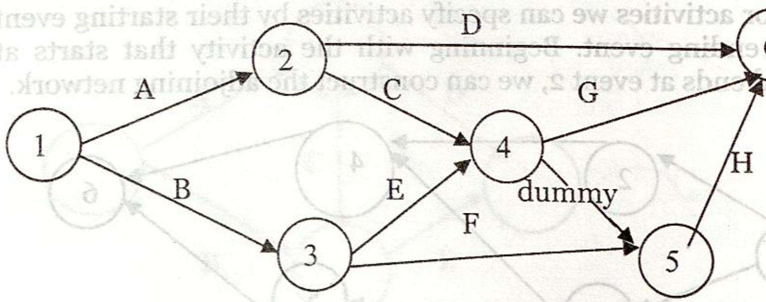
Solution

Activities A and B have no preceding activities and can commence immediately.

Activities C and D can start, once A has finished

Activities E and F can start, once B has finished. Note that activity E has been given the same head event as activity C.

The network is then completed as shown in the adjoining diagram.



Activity G follows C and E. Since activity H follows C,E and F, there is a necessity for the dummy activity. The nodes have been numbered from left to right.

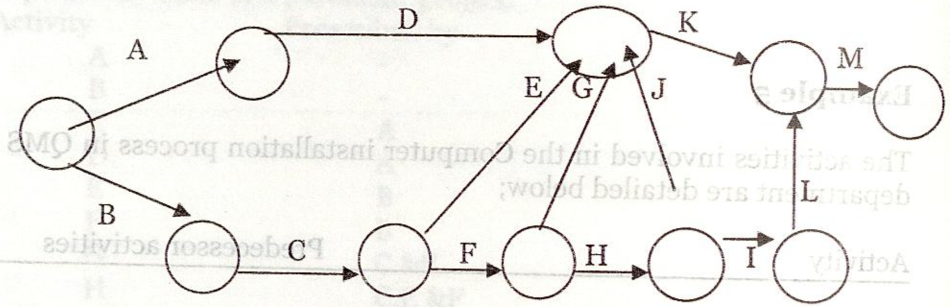
Example 5

The activities involved in the Computer installation process in QMS department are detailed below;

Activity		Predecessor activities
A	Physical preparation	-
B	Organizational planning	-
C	Personnel selection	B
D	Equipment installation	A
E	Personnel training	C
F	Detailed system design	C
G	File conversion	F
H	Establish standards and control	F
I	Programme preparation	H
J	Programme testing	I
K	Parallel operations	D,E,G,J
L	Finalize system documentation	I
M	Follow up	K,L

Draw the network for the above processes.

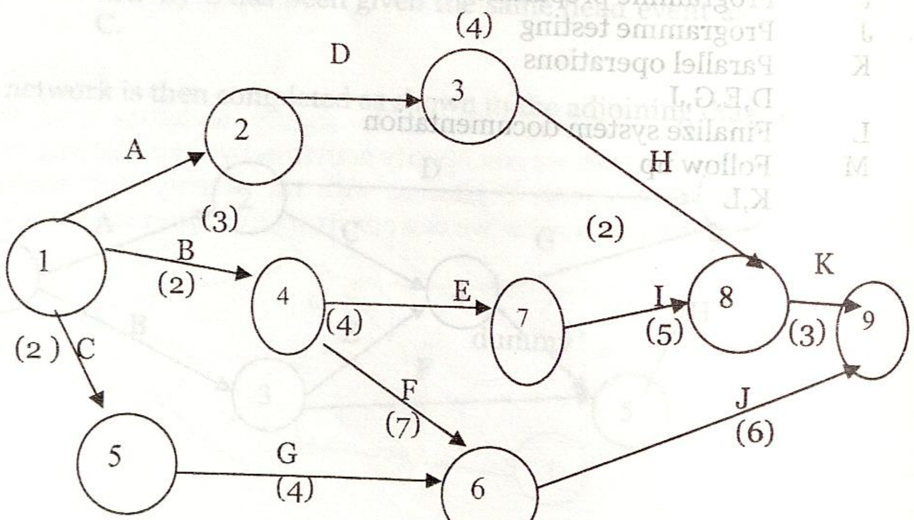
Solution



Example 6

For the following PERT diagram

i) Compute earliest event time and latest event time
ii) Critical path and total project duration
iii) Total, free and independent float for each activity



Solution

To determine earliest start time, E_j and latest finish time, L_j for each time, proceed as follows:

Forward Pass Method

$E_1 = 0$

$E_2 = E_1 + t_{1,2} = 0 + 3 = 3$

$E_3 = E_2 + t_{2,3} = 3 + 4 = 7$

$E_4 = E_1 + t_{1,4} = 0 + 2 = 2$

$E_5 = E_1 + t_{1,5} = 0 + 2 = 2$

$E_6 = Max[E_4 + t_{4,6}; E_5 + t_{5,6}] = Max[2 + 7; 2 + 4] = 9$

$E_7 = E_4 + t_{4,7} = 2 + 4 = 6$

$E_8 = Max[E_3 + t_{3,8}; E_7 + t_{7,8}] = Max[7 + 2;\ 6 + 5] = 11$

$E_9 = Max[E_8 + t_{8,9};\ E_6 + t_{6,9}] = Max[11 + 3;\ 9 + 6] = 15$

Backward Pass Method

$L_9 = E_9 = 15$ $\quad L_8 = L_9 - t_{8,9} = 15 - 3 = 12$

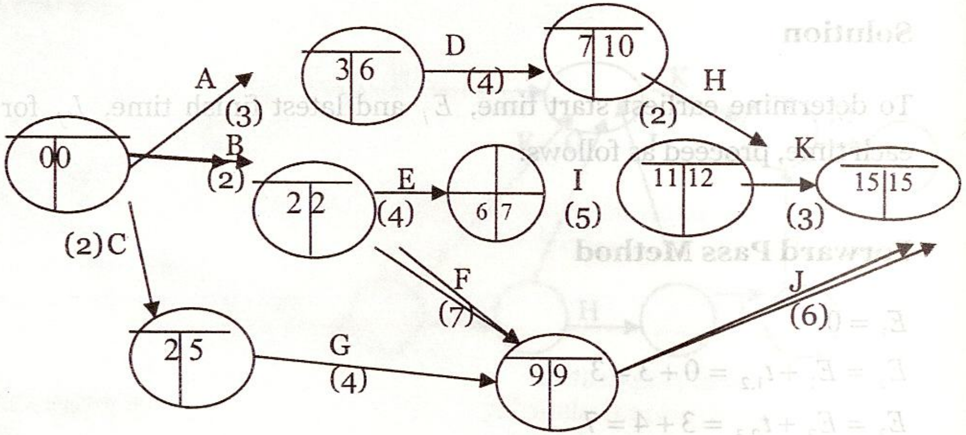
$L_7 = L_8 - t_{7,8} = 12 - 5 = 7$ $\quad L_6 = L_9 - t_{6,9} = 15 - 6 = 9$

$L_5 = L_6 - t_{5,6} = 9 - 4 = 5$ $\quad L_4 = Min[L_7 - t_{4,7}; L_6 - t_{4,6}] = Min[7 - 4; 9 -$

$L_3 = L_8 - t_{3,8} = 12 - 2 = 10$ $\quad L_2 = L_3 - t_{2,3} = 10 - 4 = 6$

$L_1 = Min[L_2 - t_{1,2}; L_4 - t_{1,4}; L_5 - t_{1,5}] = Min[6 - 3;\ 2 - 2;\ 5 - 2] = 0$

The network diagram after computations becomes;



The critical path shown by double lines in the above diagram is the longest time path throughout the network or path B- F- J, where the Earliest time values are equal to the Latest time values. Thus there is no slack time and the events must be completed exactly as scheduled to meet the completion time of fifteen (15) days.

Computation of Total, Free and Independent Floats

Activity Duration		Total Float $LS_{ij} - ES_{ij}$ or $(L_j - t_{ij}) - ES_{ij}$	Free Float = Total Float- Head Slack	Independent Float = Free float – Tail Slack
A	3	3- 0 =3	3- 3 =0	0 - 0 = 0
B	2	0- 0= 0	0- 0= 0	0 - 0 = 0
C	2	3- 0= 3	3- 3= 0	0 – 0 = 0
D	4	6-3 = 3	3-3 = 0	0 – 3 = -3
E	4	3- 2= 1	1- 1= 0	0 – 0 = 0
F	7	2- 2 = 0	0- 0 = 0	0 – 0 = 0
G	4	5- 2 = 3	3- 0 = 3	3 – 3 = 0
H	2	10- 7 = 3	3- 1 = 2	2- 3 = -1
I	5	7- 6 = 1	1- 1 = 0	0 – 1 = -1
J	6	9- 9 = 0	0- 0 = 0	0 – 0 = 0
K	3	12- 11= 1	1- 0 = 1	0 - 1 = - 1

Example 7

A project is made up of 8 jobs as follows;

Job	A	B	C	D	E	F	G	H
Duration (days)	30	36	16	14	8	23	18	20

The following sequential relationships apply:

A precede all other jobs except B and C
G must follow D
H must follow E and B and
E must follow B

a) Draw an arrow diagram for the project
b) Find the minimum duration of the project
c) What jobs lies on the critical path?
d) What is the latest starting time of job C?
e) If F were to take 36 days with all other duration unchanged, find by how much will the project duration get delayed?

Solution

The relations can be written as;

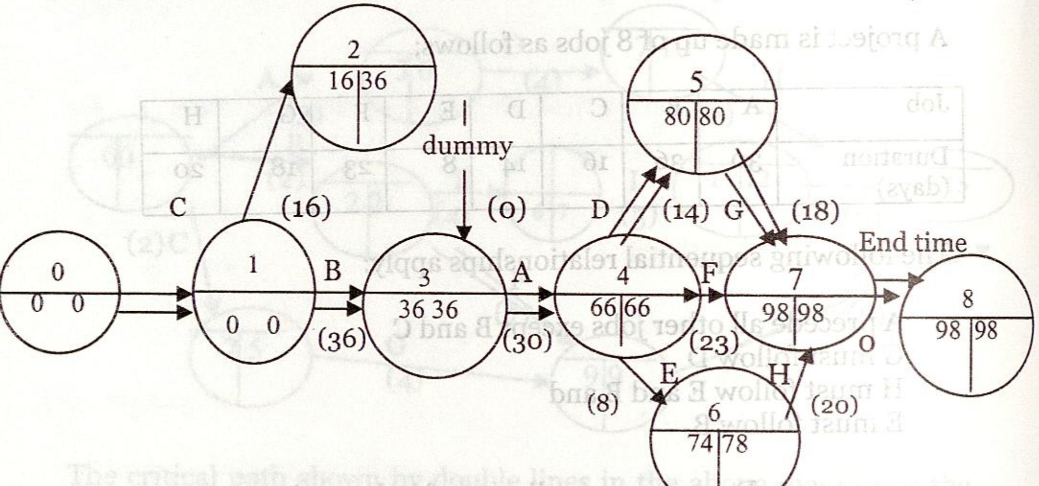
i) BC→ A→ D, E, F, G and H ii) D→ G iii) B→ H iv) B→ E

Combining (iii) and (iv) by the help of (i) we have;

v) B→ A→ E→ H and from (ii) we have C→ A→ D→ G .... (vi)

and from (i) we get A→F .... (vii).

Now, from (v), (vi) and (vii) network diagram may be written as;



Applying forward pass to find ES_{ij} and backward pass for LF_{ij} of each activity and indicate those at each node. The critical path is B– A– D– G with duration of the project as 98 days.

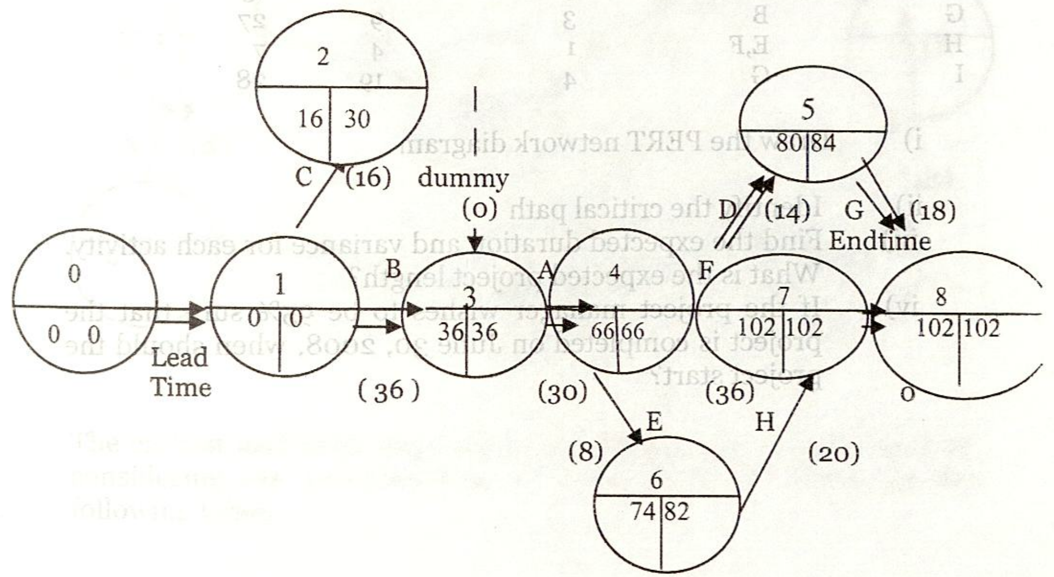
Next is the computation of scheduling times and floats;

Job	Duration	Earliest time Start	Earliest time Finish	Latest time Start	Latest time Finish	Float Total	Float Free
A	30	36	36	66	66	0	0
B	36	0	0	36	36	0	0
C	16	0	20	16	36	20	0
D	14	66	66	80	80	0	0
E	8	66	70	74	78	4	0
F	23	66	75	89	98	9	9
G	18	80 80		98	98	0	0
H	20	74	78	94	98	4	0

Note: All computations in the above table are like those in page 122. Now, the sub- critical path with 4 days or less total float is B – A –E –H of which B an A have total float 0 (zero) and E and H have total float 4 days.

i) Latest start time of C is 36 – 16 = 20 days.

ii) With the change in the duration of F as 36 days, the network will be modified as:



The duration of the project is now 102 days with critical path B –A –F

Example 8

A project has the following activities and other characteristics:

Activity	Preceding activity	Most optimistic	Most likely	Most Pessimistic
		Time estimates (in weeks)		
A	-	4	7	16
B	-	1	5	15
C	A	6	12	30
D	A	2	5	8
E	C	5	11	17
F	D	3	6	15
G	B	3	9	27
H	E,F	1	4	7
I	G	4	19	28

i) Draw the PERT network diagram

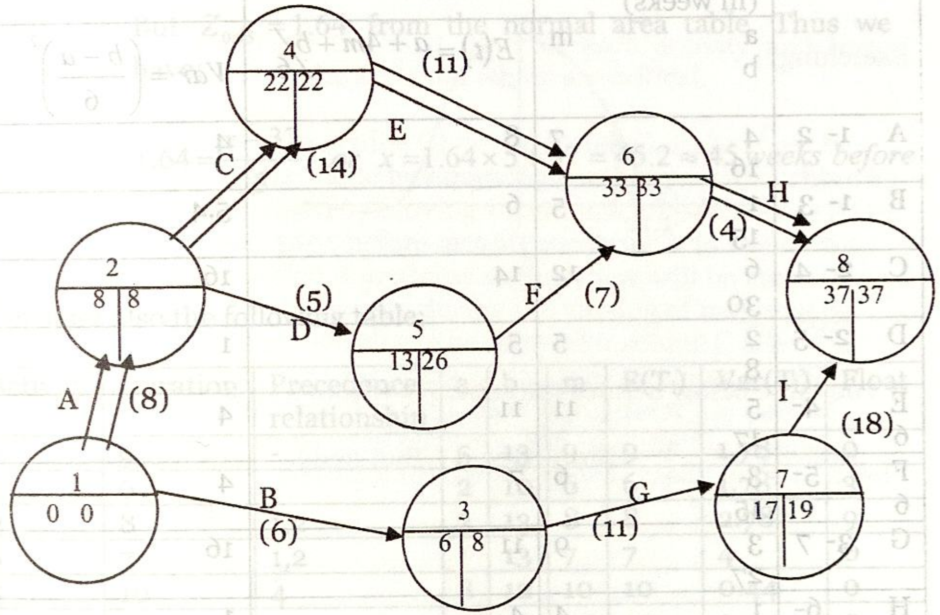
ii) Identify the critical path

iii) Find the expected duration and variance for each activity. What is the expected project length?

iv) If the project manager wishes to be 95% sure that the project is completed on June 30, 2008, when should the project start?

Solution

The network diagram for the given data is shown below;



The earliest and latest expected time for each event is calculated by considering the expected time of each activity as shown in the following table:

Calculation of expected times and variances

Activity		Time estimates (in weeks) a	m	b	Mean time $E(t) = \frac{a+4m+b}{6}$	Variance $Var = \left(\frac{b-a}{6}\right)^2$
A	1- 2	4	7	16	8	4
B	1- 3	1	5	15	6	5.4
C	2- 4	6	12	30	14	16
D	2- 5	2	5	8	5	1
E	4- 6	5	11	17	11	4
F	5- 6	3	6	15	7	4
G	3- 7	3	9	27	11	16
H	6- 8	1	4	7	4	1
I	7- 8	4	19	28	18	16

It may be observed from (i) that the critical path of the project is 1– 2– 4– 6– 7– 8, the critical activities being A,C,E and H.

The expected duration and variance for each activity is shown in the table above. The expected project length is the sum of the duration of each critical activity. i.e Expected project length = 8 + 14 + 11 + 4 = 37 weeks and the variance of the project length is the sum of the variances of each critical activity; that is;

Variance of the project length:

$\sigma^2 = 4 + 16 + 4 + 1 = 25 \text{ weeks}$

Given that $\Pr ob\left(Z \le \frac{x-\mu}{\sigma}\right) = 0.95$

But $Z_{0.95} = 1.64$ from the normal area table. Thus we have;

$1.64 = \frac{x-37}{5}$ or $x = 1.64 \times 5 + 37 = 45.2 \approx 45$ *weeks before*

Consider also the following table;

Activity	Duration	Precedence relationship	a	b	m	E(Ti)	Var(Ti)	Float
1	9	-	5	13	9	9	1.78	0
2	6	-	2	10	6	6	1.78	3
3	8	1,2	3	13	8	8	2.78	9
4	7	1,2	1	13	7	7	4	0
5	10	4	8	12	10	10	0.44	0
6	12	3,5	9	15	12	12	1	0

Computations for mean and variance becomes;

Expected Mean = E (T_i) = $\sum E(T_{ij})$ along Critical Path

= 9+7+10+12=38

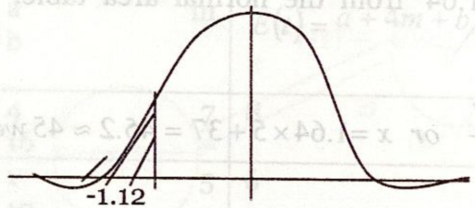
Variance =Var (T_i) =1.78+4+0.44+1=7.22

Standard deviation = $\sqrt{7.22}$ =2.69

Now, assuming that T is normally distributed, and then we can compute the probability that the project will be completed within 35 days.

$$P(T \leq 35) = P\left(\frac{T-38}{2.69} \leq \frac{35-38}{2.69}\right) = P(Z \leq -1.12) = \underline{0.13}$$

Sketching;



Exercise

1. The table below defines the activities within a small project:

Activity	Start node	End node	Completion Time
1	1	2	2
2	1	3	3
3	1	4	2
4	2	5	3
5	3	6	7
6	4	6	5
7	5	7	4
8	6	7	9
9	7	8	3

In addition to the above information we have that activity 7 can not start until activity 5 has been completed.

i) Draw the network diagram
ii) Calculate the minimum overall project completion time
iii) Calculate the float time for each activity and hence identify the activities which are critical.

2. Before a new press (machine C) can be installed, a site must be prepared for it. This is done by removing another press(machine B) to a site which is released by moving a third machine (machine A) to a new site. It is necessary, before moving machine B, to build up stocks of the parts which it prodeces so that there will be stock in hand to "carry over" the factory during the moving of machine B. There are tools in store which can be used on machine C after they have been modified, but new material handling equipment will be required. Some of this equipment ("basic equipment") is needed before machine C can be tried out but other equipment ("final equipment") is not essential until the machine is ready to be handed over to production.

Activities Involved

Prepare site for machine A	2 weeks
Prepare site for machine C	8 weeks
Move machine A	4 weeks
Move machine B	1 week
Install and wire up machine C	2 weeks
Obtain machine C	2 weeks
Try out, trouble- shoot and prepare machine C prior to hang over to production	13 weeks
Modify all tools for machine C	13 weeks
Obtain and install basic handling equipments	21 weeks
Obtain and install final handling equipments	39 weeks
Build stocks to release machine B	2 weeks

Draw the network for the above problem and determine the critical activities.

3. The table below defines the activities within a small project.

Activity	Start node	End node	Completion time (weeks)
1	1	2	2
2	1	3	4
3	2	4	7
4	3	4	3
5	3	5	7
6	4	5	3
7	5	6	4
8	4	6	6
9	6	7	2
10	4	7	7

In addition to the above information, activity five cannot start until three weeks after the end of activity one.

- Draw the network diagram.
- Calculate the minimum overall project completion time.
- Calculate the float time for each activity and hence identify the critical path.

Comment on the potential effect upon the overall project completion time (and the critical path) of:

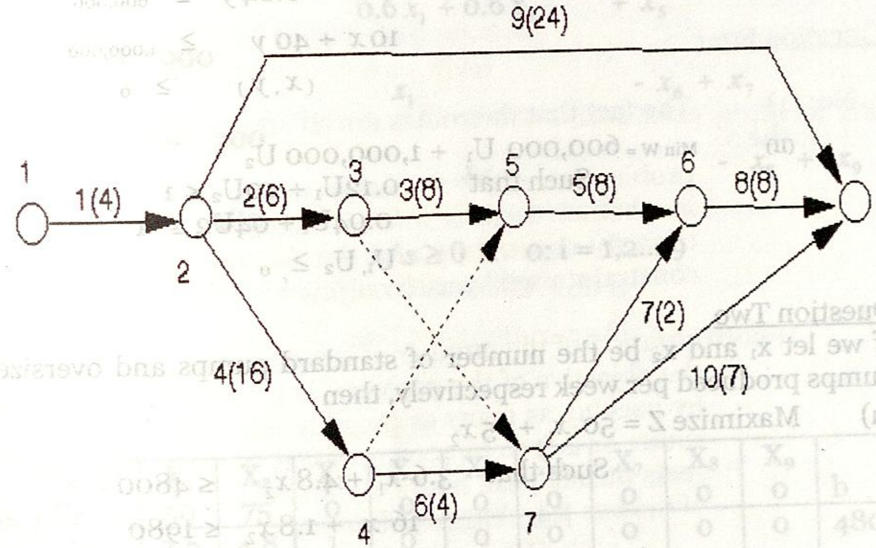
- Cutting the completion time of activity eight by three weeks?
- Increasing the completion time of activity four by two weeks?
- Cutting the completion time of activity seven by two weeks?

Will the project finish on time if at the end of six weeks the status of the project is:

- **Finished - activities one. two and four**

- In progress - activity three (five weeks to completion) and activity five (four weeks to completion)

4. For the network shown below what are the critical activities?



ANSWERS TO EXERCISES

CHAPTER TWO

<u>Question One</u>

(i) Minimize $Z = x + y$

Such that $0.12x + 0.04y \geq 600{,}000$

$10x + 40y \geq 1{,}000{,}000$

$(x, y) \geq 0$

(II) Min $W = 600{,}000\ U_1 + 1{,}000{,}000\ U_2$

Such that $0.12U_1 + 40U_2 \leq 1$

$0.04U_1 + 04U2 \leq 1$

$U_1, U_2 \geq 0$

<u>Question Two</u>

If we let x_1 and x_2 be the number of standard pumps and oversize pumps produced per week respectively, then

(a) Maximize $Z = 50x_1 + 75x_2$

Such that $3.6x_1 + 4.8x_2 \leq 4800$

$16x_1 + 1.8x_2 \leq 1980$

$0.6x_1 + 0.6x_2 \leq 900$

$x_1 \leq 300$

$x_2 \leq 180$

$x_1 \geq, x_2 \geq 0$

If we let x_3, x_4 and x_5 be slack variable for constants 1, 2, 3 respectively from (a). let also that x_6 and x_7 be surplus and artificial variables respectively for constraint 4 and x_8 and x_9 be surplus and artificial variables for constant 5 respectively; then

(b) Maximize

$Z = 50x_1 + 75x_2 + 0x_3 + 0x_4 + 0x_5 + 0x_6 - 0x_7 + 0x_8 - 0x_9$

Such that $3.6x_1 + 4.8x_2 + x_3 = 4800$

$1.6x_1 + 1.8x_2 + x_4 = 1980$

$0.6x_1 + 0.6x_2 + x_5 = 900$

$x_1 - x_6 + x_7 = 300$

$x_2 - x_8 + x_9 = 180$

$x_i\text{'s} \geq 0 \geq 0: i = 1,2....9$

(c)

X_B	C_B \ Cj	X_1 50	X_2 75	X_3 0	X_4 0	X_5 0	X_6 0	X_7 0	X_8 0	X_9 0	b
X_3	0	3.6	4.8	1	0	0	0	0	0	0	4800
X_4	0	1.6	1.8	0	1	0	0	0	0	0	1980
X_5	0	0.6	0.6	0	0	1	0	0	0	0	900
X_7	0	1	0	0	0	0	-1	1	0	0	300
X_9	0	0	1	0	0	0	0	0	-1	1	180
	Zj	0	0	0	0	0	0	0	0	0	0
	Zj -Cj	-50	-75	0	0	0	0	0	0	0	

Question Three

Minimize $850y_1 + 200y_2$

Such that $40y_1 + 5y_2 \geq 20$

$30y_1 + 10y_2 \geq 25$

$y_1, y_2 \geq 0$

Question Four

No; (1) the The fact that the routes are all straight does not make problem amenable to Linear programming. The issue is whether there is a fixed relationship between the capacity constraints and the faction of the capacity constraint used by each output.

If maximization of the number of bus-traveling population is the objective of the problem, is it to be accomplished by packing as many as possible into each bus in each route; or is it to be accomplished by offering as many seats during peak hours to maximize the convenience of the bus traveling population?

If the maximization is of the convenience factor, then the number of seats offered during rush hours may be an important element in terms of which the objective function can be framed. Given the objective function, one has to raise the question; what are the constraints on this objective: it can't be simply the total number of seats, but also their distribution which buses, which routes, at what time of the day?

If the waiting time is to be reduced, use queuing theory. If the flow of traffic is to be maximized, use network theory.

Question Five

Duality is required to determine the production that maximizes profit when the unit contributions (profits) are unavailable but the unit costs are available.

Question Six

Ignoring slacks in the Linear Programming

If the unit were y_1 and y_2, then

$y_1 = 4$ and $y_2 = 8$ gives a maximum profit of US Dollar 96.

Question Seven

The point that minimizes the objective function is

$x_1 = 8$; $x_2 = 5/3$ and a minimum cost is 380 units.

CHAPTER THREE

Question One

The assignment that will minimize cost is;

From Freighter	To Deep-Water Berth
A	3
B	4
C	5
D	1
E	2

Question Two

		Depots to be enlarged					Sup
		Q	R	S	T	dummy	
Deports to be enlarged	A	5					5
	B	3	4				4
	C		5	6			6
	D			5	3		3
	E				5	4	4
demand		3	5	5	5	4	

The minimum transportation cost is
(5 x 3) + (3x6) + (4x5) + (5x4) + (6x8) + (5x5) + (3x4) + (5x5) + (4x0) = 183 units.

Question Three (a)

Factory	Transportation Cost (USD) to Customer					
	1	2	3	4	5	6
A	12.8	13.1	14.4	15.5	13.8	14.3
B	13.2	15.6	14.5	13.4	12.8	15.0
C	14.4	15.6	12.4	15.2	13.6	12.8

(b)

From Factory	To Customer	Quality (tons)
A	1	40
A	2	20
B	4	10
B	5	60
C	2	15
C	3	25
C	4	10
C	6	30

4.

Client	Salesmen
1	D
2	A
3	C
4	B

5.

Man	Job
1	4
2	3
3	1
4	5
5	2

CHAPTER FOUR

Question One

i) 40 rolls ii) 1.25 weeks iii) Tsh. 40,000

Question Two

b) i) Optimum run size is 749.8 ≈ 750 units
ii) Cycle length = 0.5 years or 6 months

Question Three

a) Q = 2927.7 ≈ 2928
b) TC = Tsh. 12,808,688.46
c) 51.2 times a year

Question Four
i) EOQ = 53,665.63 units
ii) 0.00447 years
iii) Tshs 6,005,366,563

CHAPTER FIVE

Question One

i) $p = \frac{1}{6}$ or 0.7 ii) 5 customers iii) 4.05 customers iv) 15 minutes or $\frac{1}{4}$ hours

v) 12.15 minutes or 0.20 hours

Question Two

a. A – Probability distribution of the arrivals
B – Probability distribution of the service
C- Number of servers
D- Queuing discipline
E- Buffer size/ capacity of the queue system

b) i) D/M/4 ii) M/M/1

Question Three

a) Arrival rate is 56 per hour, service rate is 101.02 per hour
b) 0.68 customers
c) 0.022 hours

Question Four

i) $\frac{3}{4}$ planes ii) $\frac{1}{8}$ hours

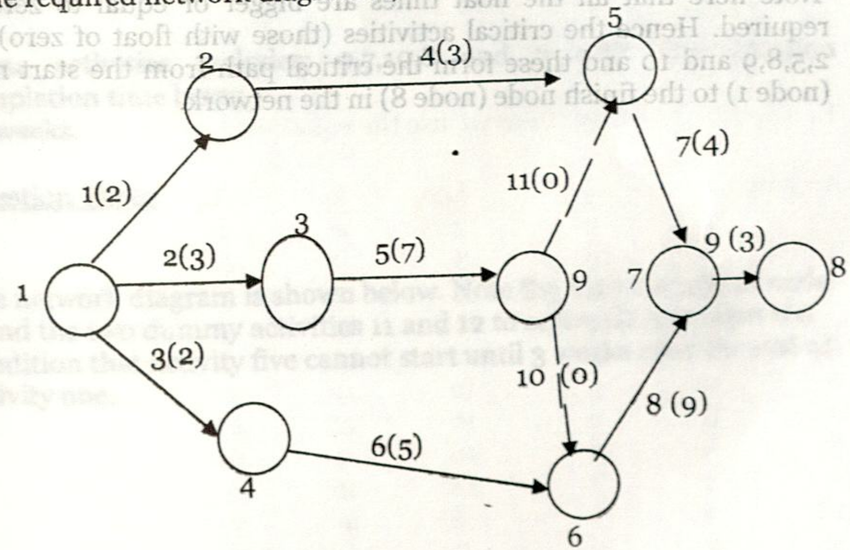
Question Five

i) $p = \frac{4}{9}$ ii) $\frac{8}{9}$ customers

CHAPTER SIX

Question One

i) The required network diagram is;

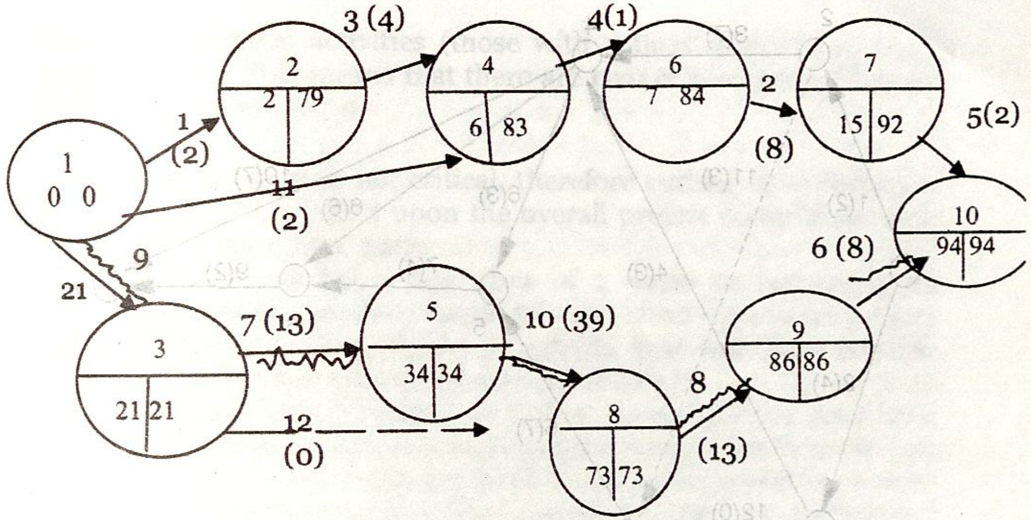


ii) 22 weeks

iii) Activity	i	j	L_j	E_i	T_{ij}	F_{ij}
1	1	2	12	0	2	10
2	1	3	3	0	3	0
3	1	4	5	0	2	3
4	2	5	15	2	3	10
5	3	9	10	3	7	0
6	4	6	10	2	5	3
7	5	7	19	10	4	5
8	6	7	19	10	9	0
9	7	8	22	19	3	0
10	9	6	10	10	0	0
11	9	5	15	10	0	5

Note here that all the float times are bigger or equal to zero as required. Hence the critical activities (those with float of zero) are 2,5,8,9 and 10 and these form the critical path from the start node (node 1) to the finish node (node 8) in the network.

Question Two

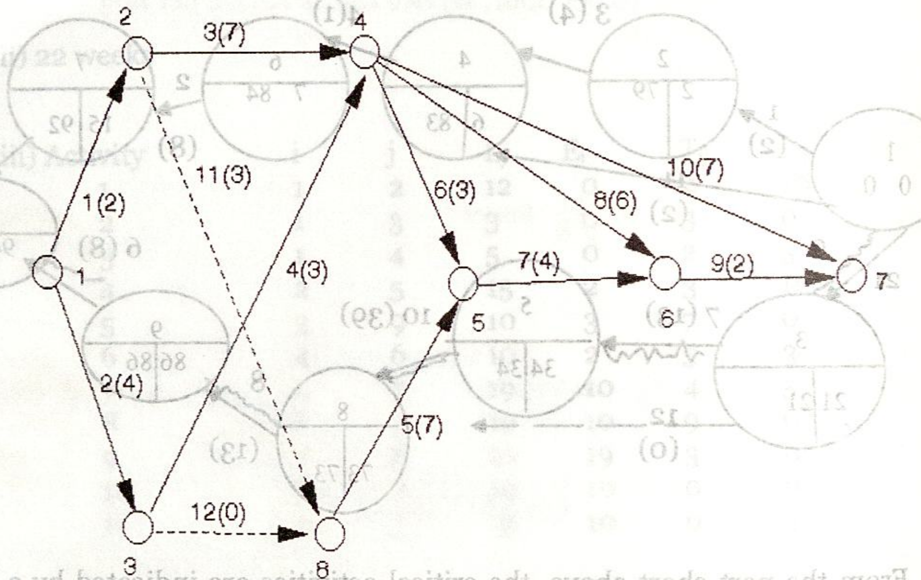


From the pert chart above, the critical activities are indicated by a rough line.

These activities includes; 9,7,10,8 and 6 with the duration completion time being 94 weeks.

Question Three

The network diagram is shown below. Note the introduction of node 8 and the two dummy activities 11 and 12 to correctly represent the condition that activity five cannot start until 3 weeks after the end of activity one.



The minimum overall project completion time is 18 weeks.

To calculate the float times we use the equation $F_{ij} = L_j - E_i - T_{ij}$ to get

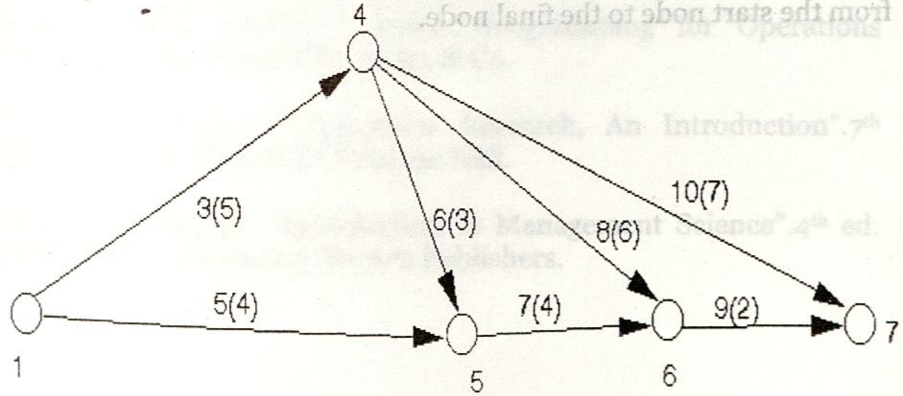
Activity	i	j	L_j	E_i	T_{ij}	F_{ij}
1	1	2	2	0	2	0
2	1	3	5	0	4	1
3	2	4	9	2	7	0
4	3	4	9	4	3	2
5	8	5	12	5	7	0
6	4	5	12	9	3	0
7	5	6	16	12	4	0
8	4	6	16	9	6	1
9	6	7	18	16	2	0
10	4	7	18	9	7	2
11	2	8	5	2	3	0
12	3	8	5	4	0	1

Note here that all float times are >= 0 as required.

Hence, the critical activities (those with a float of zero) are 1, 3, 5,6,7,9 and 11. This means that there are **two** critical paths, namely 1-11-5-7-9 and 1-3-6-7-9.

- activity eight is not critical, therefore cutting its completion time has no effect upon the overall project completion time or on the critical paths
- activity four has a float time of 2 weeks so increasing its completion time by 2 weeks does not effect the overall project completion time. However, activity four will then become critical so the critical paths will be effected.
- activity seven is critical so cutting its completion time by 2 weeks may reduce the overall project completion time. In fact as activity seven appears in *all* (both) critical paths we can be sure that the overall project completion time will be reduced by *at least* one time unit (week). The critical paths may, or may not, be effected.

After six weeks the new network diagram is shown below.



The earliest start time calculation is

$E_1 = 0$ (by definition)
$E_4 = E_1 + T_{14} = 0 + 5 = 5$
$E_5 = \max [E_1 + T_{15}, E_4 + T_{45}] = \max [0 + 4, 5 + 3] = 8$
$E_6 = \max [E_5 + T_{56}, E_4 + T_{46}] = \max [8 + 4, 5 + 6] = 12$
$E_7 = \max [E_6 + T_{67}, E_4 + T_{47}] = \max [12 + 2, 5 + 7] = 14$

Hence, the minimum overall completion time for the remaining part of the project is 14 weeks.

As 6 weeks have already elapsed this means that we cannot finish the complete project before week 20, i.e. so far we have slipped 2 weeks upon the original completion time of 18 weeks and there is no possibility of the project finishing upon time.

Question Four

The critical activities are 1, 4, 5, 8 and these (together with the dummy activity between node 4 and node 5) form the critical path from the start node to the final node.

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