PRICING BARRIER OPTIONS WHEN THE DYNAMICS OF THE PRICES ARE DRIVEN BY THE MEAN REVERTING PROCESS

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PRICING BARRIER OPTIONS WHEN THE DYNAMICS OF THE PRICES ARE DRIVEN BY THE MEAN REVERTING PROCESS

By

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A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science (Mathematical Modelling) of the University of Dar es Salaam

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CERTIFICATION

The undersigned certify that they have read and hereby recommend for acceptance by the University of Dar es Salaam a dissertation entitled:***Pricing Barrier Options When the Dynamics of the Prices are Driven by the Mean Reverting Process***, in fulfillment of the requirements for the degree of Master of Science (Mathematical Modelling) of the University of Dar es Salaam.

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DECLARATION
AND
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I, **Masoud Komunte**, declare that this dissertation is my own original work and that it has not been presented and will not be presented to any other University for a similar or any other degree award.

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DEDICATIONS

I would sincerely like to dedicate this dissertation to my late father AMRI MOHAMED KOMUNTE whose invaluable contributions to my achievements will never be neglected. I shall always remember you dad, may your soul rest in eternal peace. Amen.

ABBREVIATIONS

CBOE	Chicago Board Options Exchange
CBOT	Chicago Board of Trade
DGBM	Drifted Geometric Brownian Motion
GBM	Geometric Brownian Motion
HAM	Homotopy Analysis Method
MRL	Mean Reverting Lognormal
MRLM	Mean Reverting Lognormal Model
PDE	Partial Differential Equation
SDE	Stochastic Differential Equation
U.S.	United States

ABSTRACT

This dissertation considers a problem of pricing barrier options when the dynamics of the asset prices $(X(t))$ are driven by the mean reverting process, the market/asset price $X(t)$ is obtain from mean reversion model and a Black-Scholes PDE model for pricing barrier options under mean reversion model is obtained upon using It$\hat{o}$ formula. Through the Homotopy Analysis Method (HAM) the price of the chosen barrier option (up$-$and$-$out European call) that satisfies the Black-Scholes PDE model was determined. Thus, through HAM we can determine approximated prices of barrier options when the dynamics of the prices are driven by the mean reverting process (Liao, 2004).

Lastly the analysis is conducted to observe the behaviour of the option price when value of one parameter increases while the value of the other two parameters remain constant. The analysis shows that the option price tends to increase with the increase of the value of the parameter for the case of volatility and degree of mean reversion while for interest rate the option price decreases when interest rate increases. In all cases it is observed that early exercise is better than late exercise to owner of the option since the option price tends to decrease as time increases also to minimize risk owner of the option should exercise the option when the volatility of the market become large.

It is recommended that in future, areas of interest for research related to this study are; first, finding the option price by using direct integration after obtaining a reflection principle which is useful in determining the joint distribution of the It$\hat{o}$ integral and secondly, finding the price which is a closed form solution by using Laplace transform.

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CHAPTER ONE

INTRODUCTION

1.1 General Introduction

An option is a security giving the right but not obligation to buy or sell an asset at a fixed price, subject to certain conditions, within a specified period of time. The act of making this transaction is referred to as exercising the option. The fixed price is known as the strike price, and the given date is known as the expiration date. A call option gives the right but not obligation to buy the asset, a put option gives the right but not obligation to sell the asset. An American option is the one that can be exercised at any time up to the date the option expires while European option is the one that can be exercised only on a specified future date (Shreve, 2004).

If the holder of the option decides to exercise the option then the other trader also known as writer of the option, is obliged to exercise the option regardless of what loss he/she is going to encounter. Since the option favours its holder to decide to exercise or not to exercise the option, the holder must pay a premium for the rights the option confers on him. This payment is also referred to as the option value (Shreve, 2004).

Options had been traded for centuries, but they remained obscure financial instruments until the introduction of a listed options exchange in 1973 at the Chicago Board Options Exchange (CBOE). Since then, options trading have enjoyed an expansion never done before in American securities markets.

According to Cox and Rubinstein (1979), option pricing theory has an illustrious and long history, but it also underwent a nice revolutionary change in 1973. At that time, Myron Scholes and Fischer Black presented the first completely satisfactory equilibrium option pricing model. They provided pricing formulae for European options on a single stock where the stock price follows a geometric Brownian motion (GBM).

Therefore in order to get a clear picture of what will be discussed in this study, the following are some definitions of the terms related to options.

1.1.1 Derivative

Definition 1.1 (Hull, 2002): A derivative is a security whose value depends on (or derives from) the values of other, more basic underlying security (asset). For example, a stock option, is a derivative whose value is dependent on the price of a stock. Common types of derivatives are options, forwards and futures contracts and swaps.

Definition 1.2 (Baxter *et al.*, 1996): Underlying security (asset) refers to any market security such as bond, stock, share, commodity and currencies.

There are various types of options which includes European options, American options, Asian options, lookback options and Barrier options. In this study our main focus is on barrier options.

1.1.2 Barrier Options

Definition 1.3 (Hull, 2002): A barrier option is an exotic (path dependent) option where the payoff depends on whether the underlying asset's price reaches a certain level (barrier L) during a certain period of time.

We have two cases which are knock-in options and knock-out options. A knock-in option comes into existence only when the underlying asset price reaches a barrier L while a knock-out option ceases to exist when the underlying asset price reaches a certain barrier L.

1.1.2.1 Categorizing barrier options

Barrier options specify a stock price level, L say, such that the option pays (knocks in) or do not pay (knocks out) according to whether or not level L is attained, from below (up) or above (down). There are thus four possibilities namely "up and in", "up and out", "down and in" and "down and out" (Bingham *et al.*, 2004).

Together with European call or put, simply there are eight standard types of barrier options as illustrated in table 1.1 below.

Table 1.1: Categorization of Barrier Options

Call				Put			
Up		Down		Up		Down	
In	Out	In	Out	In	Out	In	Out

For example, down and out European call (D&O call) is the European call that is knocked out (or "deactivated") if the underlying hits a barrier L from above before expiration.

If a barrier L is smaller than the present asset value S_0. Payoff of the D&O call is

$$C_T^{D\&O} = \begin{cases} (S(t)-K)^{+} \ \text{ if } \ S(t) > L \text{ for all } \ t \leq T \\ 0 \ \text{ if } \ S(t) < L \ \text{ for at least one } t \leq T \end{cases} \qquad \text{where K is the strike price.}$$

The simplest case is by using a switch which takes the value one or zero, now the Payoff of the D&O Call is given by $C_T^{D\&O} = (S(T)-K)^{+} I_{\{Min\ S(t) \geq L, 0 \leq t \leq T\}}$ where K is the strike price.

Since the barrier option is down and out, when the price goes down the barrier L then it is “knocked-out” that is its payoff is zero and when the price goes up of the barrier L then “it stays active” that is its payoff is $S(T)-K$. Also when the price stays within the barrier L that is $K = L$ the payoff is also zero. The figure 1.1 below shows illustration on these prices.

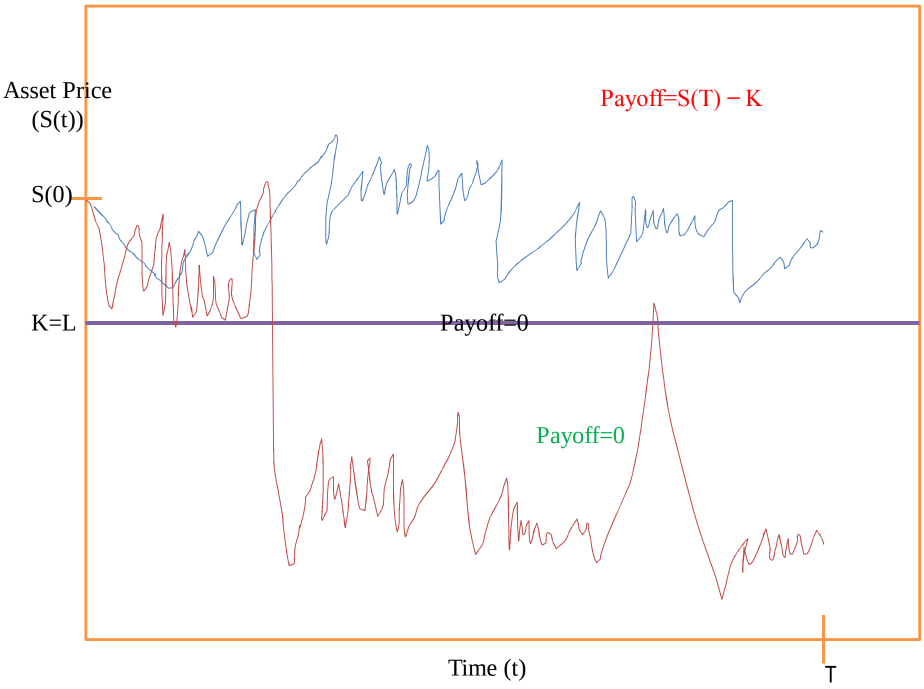

Figure 1.1: Prices for Down and out call

1.1.3 Mean Reversion

Definition 1.4 (Espen, 2006): Mean reverting (mean reversion) is a mathematical concept sometimes used for asset investing, the essence of the concept is the assumption that both an asset's high and low prices are temporary and that an asset's price will tend to move to the average price over time.

Mean reversion involves first identifying the trading range for an asset, and then computing the average price using analytical techniques. When the current market price is less than the average price, the asset is considered attractive for purchase, expecting that the price will soon rise. When the current market price is above the average price, the market price is expected to fall. That is to say, deviations from the average price are expected to revert to the average.

The intention of this study was to price barrier options when the dynamics of the prices are driven by the mean reverting process. The Black-Scholes PDE model for pricing barrier options was formulated and finally prices of the barrier options that satisfies the Black-Scholes PDE model were determined upon using homotopy analysis method (HAM) which is a very useful method of solving nonlinear differential equations.

1.2 Statement of the Problem

Most of the studies on pricing barrier option use Geometric Brownian model (GBM) which is always Markovian, this implies that the future behaviour of the price of the barrier option depends only on the present state when it is known and not its past state (Øksendal, 2003). However in real life situations price of the barrier option not only depend on the present state but also on the past states. Therefore price of the barrier option can better be modeled by geometric mean reversion model (Hui and Lo, 2006).

This study therefore intends to establish a method of pricing barrier options when dynamics of the prices are driven by the mean reverting process so that it can be observed how mean reversion model affect the pricing of barrier option and the volatility of an asset price. HAM will be used to find the approximated price of the barrier option, barrier options were studied because they are attractive to market participants since they are less expensive than the corresponding regular options and other exotic options.

1.3 Research Objectives

1.3.1 Main Objectives

The general objective of the study was to determine prices of barrier options when the dynamics of the prices are driven by the mean reverting process.

1.3.2 Specific Objectives

The specific objectives of this study were as follows:

(i) To formulate Black-Scholes PDE model for pricing barrier options under mean reversion dynamics.

(ii) To determine the prices of the barrier options that satisfies the Black-Scholes PDE model.

(iii) To determine the impact of the embedded parameter on the price of the barrier option.

1.4 Significance of the Study

The significance of this study are:

(i) The study will have a great contribution to the world of mathematics of finance with valuable knowledge on Mean reversion models and barrier options.

(ii) To the investors and decision makers, this study will help them in determining the advantage of using barrier option as one among other path dependent option rather than using normal European call or put in maximizing the profit.

(iii) To researchers within the field of mathematics of finance, it will be a benchmark or a reference toward further researches on barrier options with mean reversion model.

1.5 Research Methodology

The underlying security (asset) price is given by the mean reverting geometric Brownian motion in stochastic differential equation (SDE) of the form

$$dX(t) = k(\rho - \ln X(t))X(t)dt + \sigma X(t)d\tilde{B}(t).$$

where k, ρ and σ are positive constants in which

- ⇒ k is a constant that governs the degree of mean reversion of the process. When k is high it implies a strong reversion of the process and vice versa.
- ⇒ ρ is the long run equilibrium level of an asset price also known as interest rate.
- ⇒ σ is an instantaneous standard deviation of the relative change also known as volatility.

$\tilde{B}$ is a Brownian motion under the risk neutral measure Q given by the Girsanov theorem, first the It$\hat{o}$ formula process was used directly to obtain the market/asset price $X(t)$ from the given SDE, then by using martingale property of It$\hat{o}$ integral and iterated conditioning argument the dt term was equated to zero. As a result Black-Scholes PDE model for pricing barrier options under mean reversion model was formulated and the price of the barrier options that satisfies the Black-Scholes PDE model was determined by using homotopy analysis method (HAM) with an initial guess $u_0(x)$ that satisfied the boundary conditions.

Thereafter an analysis on the parameters k, ρ and σ of the price of the barrier option obtained is conducted to study the behaviour of the price when one parameter varies keeping the other parameters constant. As a result, plots are drawn for each case and reasonable conclusion is extracted from those plots.

Finally the summary of the work including the advantages of pricing barrier options by using mean reversion model are stated and recommendations for the future work is given.

CHAPTER TWO

LITERATURE REVIEW AND THEORETICAL BACKGROUND

2.1 Introduction

A detailed survey on previous works provides the needful information for the study. In this chapter we review relevant literature, definitions, theories and results in connection to pricing of barrier options when the dynamics of the prices are driven by the mean reverting process.

2.2 Literature Review

Several researches have been conducted and a lot of articles, papers and books have been written with the aim of studying options, some of them dealing with the knowledge of pricing options by using different models.

Shreve (2004) studied pricing of barrier option by using Geometric Brownian Motion (GBM) in which he used the Drifted Geometric Brownian Motion (DGBM) together with a new Geometric Brownian Motion (GBM) defined as the maximum value of the Drifted Geometric Brownian Motion (DGBM). He managed to find the price of the up and out call (U&O Call) using these Multiple Brownian Motions and Black-Scholes-Merton equation.

Borovkov and Novikov (2002) discussed an approach of calculating expectations of a specific form used for the pricing of derivative assets. They showed that in the case of vanilla, the expectations can be found by simply integrating the respective moment generating function with a certain weight. In situations that correspond to barrier options, they suggested to carry out one more integration. Their intention was to draw attention on barrier options and to extend a new and interesting tool for pricing barrier options. They found out that if the Laplace transforms of their random variables can be computed, then the pricing problem can be reduced to calculating a single Fourier integral (in the case of vanilla options) or inverting a Fourier transform (in the case of a barrier option) although they are not able to extend the approach to more general for the case of barrier options.

Koijen *et al.*, (2006) focused on an investor with a medium term horizon of up to five years. In these horizons, both return continuation (momentum) and mean reversion are of central importance in the asset allocation problem. They introduced a continuous time model that captures these two predictability features of stock market returns. Their model predicted that hedging demands are negative for short to medium-term investors, and that the total allocation to stocks does not increase monotonically with the investor's horizon. Moreover, they used Campbell and Viceira (1999) and Campbell *et al.*, (2003) to model Mean-reversion in stock returns via persistent financial ratios as the dividend yield or the price earnings ratio predicting future stock returns. They finally found out that the value of hedging is substantial at short investment horizons, but remains constant up to 5 to 6 years. At that point, mean-reversion becomes dominating, also the value of hedging starts gradually to increase.

Spierdijk and Bikker (2012) in their paper discussed the implications of mean reversion in stock prices for long-term investors such as pension funds. In their work, they started with a general definition of a mean reverting process and explained how mean reversion in stock prices can be related to mean reversion in stock returns. Subsequently, they showed that mean reversion makes stocks less risky for investors with long investment horizons. They considered a mean−variance efficient investor and show how mean reversion in stock prices affects such an investor's optimal portfolio weights. Finally, they discussed the implications of their findings for the investment decisions of long-term investors.

Fusari *et al.*, (2010) studied the convergence properties of various types of options including barrier options. They used numerical based approach of pricing barrier options with a suitable transition probability adjustment to demonstrate an increased convergence rate for the standard Cox-Ross-Rubinstein (1979) binomial tree model applied to barrier options. For options whose strike price is close to the barrier they were able to obtain numerical results where other models failed, although convergence tends to be slow, they were able to calculate reasonable approximations to the analytical options price.

Kou (2003) in his paper studied discrete and continuous monitoring of the barrier. He commented that most models assume continuous monitoring of the barrier, however in practice a lot of barrier options traded in markets are discretely monitored. Different from their continuous counterparts, where there is essentially no closed form solution available, also even numerical pricing is difficult. His paper extended an approximation by Broadie, Glasserman and Kou (1997) for discretely monitored barrier options by covering more cases and giving a simpler proof. The techniques used came from sequential analysis, particularly Siegmund and Yuh (1982) and Siegmund (1985).

Papanicolauo *et al.*, (2000) presented a derivative pricing and estimation methodology for a class of stochastic volatility models. From their Empirical analysis they observed that volatility reverts slowly to its mean, but it is fast mean-reverting when looked at over the time scale of a derivative contract (many months). This motivated an asymptotic analysis of the partial differential equation satisfied by derivative prices, which considerably simplified the estimation procedure.

Wong and Lau (2007) in their paper developed a path-dependent currency option pricing framework in which the exchange rate follows a mean-reverting lognormal process. They derived analytical solutions for barrier options with a constant barrier, turbo warrants and lookback options. As the analytical solutions are obtained using a Laplace transform, their study numerically showed that the solution implemented with a numerical Laplace inversion is efficient and accurate. Also the pricing behaviour of path-dependent options with mean reversion is contrasted with the Black-Scholes model.

Smith (2010) in his paper commented that mean reverting processes are widely used to model interest rates, and are of particular use to those modelling commodities. Also he stated that the most popular model is the Ornstein and Uhlenbeck model ($O-U$ model) which is a mean reverting linear model, also known as the Vasicek process. He discussed the model briefly, including Matlab code to simulate the process. He discussed the estimation of the parameters, in particular the difficult of estimating the speed-of-mean-reversion parameter. Again, he included extensive Matlab code for parameter estimation.

Hui and Lo (2006) employed a mean-reverting lognormal model (MRLM) to study the pricing behaviour of options with a soft barrier which is a deterministic function of time that is specifically constructed to match with the coefficients of the governing partial different equation of the barrier option. They found that the parameters in the MRL model have a material impact on the valuation and hedging parameters of barrier options. They also simulated the fixed barrier as a slowly fluctuating soft barrier with small oscillating amplitude by tuning a parameter in the model and finally the approximated option prices are then obtained.

2.3 Useful tools for financial options

2.3.1 Stochastic process

Suppose we observe some characteristic of a system at discrete points in time (labeled $0,1,2,...$). Let $X(t)$ be the value of the system characteristic at time t. In most situations, $X(t)$ is not known with certainty before time t and may be viewed as a random variable. A discrete-time stochastic process is simply a description of the relation between the random variables $X(0),X(1),X(2),...$

Definition 2.3 (Charles and Van der Weide, 2011):

A stochastic process is a sequence of events in which the outcome at any stage depends on some probability.

2.3.2 Markov Process

Definition 2.4 (Wikipedia, 2013):

In probability theory and statistics, a Markov process or Markoff process, named after the Russian mathematician Andrey Markov, is a stochastic process that satisfies the Markov property. A process satisfies the Markov property if one can make predictions for the future of the process based solely on its present state just as well as one could knowing the process's full history. That is conditional on the present state of the system, its future and past are independent.

Thus, a Markov process is a particular type of stochastic process where only the present value of a variable is relevant for predicting the future. The past history of the variable and the way that the present has emerged from the past are irrelevant, Markov process can be thought of as 'memoryless process'.

2.3.3 Quadratic Variation

Definition 2.5 (Øksendal, 2003):

A function $f(t)$ is said to have quadratic variation if, over the closed interval $[a,b]$, there exists an M such that $(f(t_1)-f(a))^2+(f(t_2)-f(t_1))^2+....+(f(b)-f(t_n))^2 \leq M$ for all partitions $a=t_0<t_1<t_2<.....<t_n<t_{n+1}=b$ of the interval.

2.3.4 Martingale Property

Let us define some important terminologies before we define martingale property.

Definition 2.6 (Shreve, 2004):

Let Ω be a nonempty set, and let $\mathcal{F}$ be a collection of subsets of Ω . We say that $\mathcal{F}$ is a $\sigma-$algebra (called a $\sigma-$field by some authors) provided that:

(i) The empty set $\emptyset$ belongs to $\mathcal{F}$.

(ii) Whenever a set A belongs to $\mathcal{F}$, its complement A^c also belongs to $\mathcal{F}$.

(iii) Whenever a sequence of sets $A_1, A_2, ...$ belongs to $\mathcal{F}$, their union $\bigcup_{n=1}^{\infty} A_n$ also belongs to $\mathcal{F}$.

Definition 2.7 (Shreve, 2004):

Let Ω be a nonempty set. Let T be a fixed positive number, and assume that for each $t \in [0,T]$ there is a $\sigma-$algebra $\mathcal{F}(t)$. Assume further that if $s \leq t$, then every set in $\mathcal{F}(s)$ is also in $\mathcal{F}(t)$. Then we call the collection of a $\sigma-$algebra $\mathcal{F}(t)$, $0 \leq t \leq T$, a filtration. A filtration tells us the information we will have at future times.

Definition 2.8 (Shreve, 2004):

Let X be a random variable defined on a nonempty sample space Ω . Let $\mathcal{G}$ be a $\sigma-$algebra of subsets of Ω. If every set in a $\sigma-$algebra generated by X denoted by $\sigma(X)$ is also in $\mathcal{G}$, we say that X is $\mathcal{G}-$measurable.

Definition 2.9 (Shreve, 2004):

Let Ω be a nonempty sample space equipped with a filtration $\mathcal{F}(t)$, $0 \leq t \leq T$. Let $X(t)$ be a collection of random variables whereby $t \in [0,T]$. If for each t the random variable X(t) is $\mathcal{F}(t)-$measurable then we say this collection of random variables is an adapted stochastic process .

Now let us define the martingale property.

Definition 2.10 (Shreve, 2004):

A stochastic process $X(t)$ is called an $\mathcal{F}(t)-$martingale if the following conditions holds:

(i) $X(t)$ is adapted to the filtration $\{\mathcal{F}(t)\}_{t\geq 0}$.

(ii) For all t then $E[|X(t)|] \leq \infty$.

(iii) For all s and t with $s \leq t$ the following relation holds:

$E[|X(t)|\mathcal{F}(s)|] = X(s), \ \ 0 \leq s \leq t.$

2.3.5 Standard Brownian motion

Definition 2.11 (Shreve, 2004):

Suppose $B(t)$ is a random variable whose value tells us the value of some quantity of interest at time t. For example, $B(t)$ can be an asset price. The collection of random variables $B(t) : 0 \leq t < \infty$ is said to be a standard Brownian motion if the following properties holds:

(i) $B(0) = 0$.

(ii) For all $t \geq 0$ and $0 = t_0 < t_1 < t_2 < .... < t_n$ then $B(t_1) = B(t_1) - B(t_0), B(t_2) - B(t_1), B(t_3) - B(t_2), ..., B(t_n) - B(t_{n-1})$ are independent increments.

(iii) $E[B(t_{i+1}) - B(t_i)] = 0$ and $Var[B(t_{i+1}) - B(t_i)] = t_{i+1} - t_i$.

Two popular related processes are the arithmetic Brownian motion and the geometric Brownian motion.

Definition 2.12 (Shreve, 2004):

The arithmetic Brownian motion is given by

$$X(t) = X(0) + \mu t + \sigma B(t). \tag{2.1}$$

Definition 2.13 (Shreve, 2004):

The geometric Brownian motion is given by

$$X(t) = X(0)\exp\left[(\alpha - \frac{1}{2}\sigma^2)t + \sigma B(t)\right]. \tag{2.2}$$

This is the asset price model mostly used in the Black-Scholes market. The constant α is called the appreciation rate and σ is called the volatility coefficient.

The geometric Brownian motion is also given in stochastic differential form as:

$$dX(t) = \alpha X(t)dt + \sigma X(t)dB(t). \tag{2.3}$$

From equation (2.3) above, as Shreve (2004) did, by consideration of quadratic variation it can be shown that:

$$dB(t)dB(t) = dt,\ dB(t)dt = dtdB(t) = 0 \text{ and } dtdt = 0. \tag{2.4}$$

These result are due to the fact that the amount of quadratic variation Brownian motion accumulates in an interval is equal to the length of the interval, regardless of the path along which we do the computation (Shreve, 2004).

Note that, $B(t)$ is a martingale with respect to the filtration $\mathcal{F}(t)$. That is $E\left[B(t)|\mathcal{F}(s)\right] = B(s)$.

2.3.6 It$\hat{o}$ Process

Definition 2.14 (Lawler, 2006):

Let $B(t), t \geq 0$ be a Brownian motion and let $\mathcal{F}(t)$ be the associated filtration. An It$\hat{o}$

process is the stochastic process of the form:

$$X(t) = X(0) + \int_0^t \phi(s)dB(s) + \int_0^t \Theta(s)ds. \tag{2.5}$$

Where $X(0)$ is a non random and $\phi(s)$ and $\Theta(s)$ are adapted stochastic process.

2.3.7 It*ô* Formula

Theorem 2.3 (Lawler, 2006):

Let $X(t), t \geq 0$ be an It*ô* process and let $f(t,x)$ be a function for which the partial derivatives $f_t(t,x)$ the first derivative of $f(t,x)$ with respect to time t, $f_x(t,x)$ the first derivative of $f(t,x)$ with respect to x and $f_{xx}(t,x)$ the second derivative of $f(t,x)$ with respect to x are defined and continuous (we also refer to these conditions as compact support), then for every $t \geq 0$ we have the two dimension It*ô* formula:

$$df(t,X(t)) = f_t(t,X(t))dt + f_x(t,X(t))dX(t) + \frac{1}{2}f_{xx}(t,X(t))dX(t)dX(t). \tag{2.6}$$

In the case that $X(t) = (X_1(t), X_2(t), X_3(t), ..., X_n(t))$ is an n-dimensional It*ô* process such that: $X_i(t) = X_i(0) + \int_0^t \sum_{j=1}^n \phi(s)_{ij} dB_i(s) + \int_0^t \Theta_i(s)ds$; $X_i(0) = x_i$.

Then if $X(t) = (X_1(t), X_2(t), X_3(t), ..., X_n(t))$ we have a multi dimension It*ô* formula of the form:

$$\begin{aligned} df(t,X(t)) = & \frac{\partial f}{\partial t}(t,X(t))dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(t,X(t))dX_i(t) \\ & + \frac{1}{2}\sum_{i=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(t,X(t))(\sigma\sigma^T)dX_i(t)dX_j(t) \end{aligned} \tag{2.7}$$

where $\sigma = \begin{pmatrix} \sigma_{11} & ... & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & ... & \sigma_{nn} \end{pmatrix}$.

2.3.8 Risk-Neutral Measures

Definition 2.15 (Investopedia, 2013):

A Risk-neutral measure is a theoretical measure of probability derived from the assumption that the current value of financial assets is equal to their expected payoffs in the future discounted at the risk-free rate. Another assumption made is that there is an absence of arbitrage. The term derives its name from the fact that all financial assets have the same expected rate of return that is the risk-free rate. The risk-neutral measure is also known as equivalent martingale measure or $Q-$measure.

Thus, a probability measure Q is risk-neutral measure if and only if the discounted price process $X(t)$ is a martingale under Q, that is $E^{Q}\left[X(t)|\mathcal{F}(u)\right] = X(u)$, where E^{Q} denotes the expectation under Q.

2.3.9 Feynman-Kac Formula

Let $f(t,x) \in C_0^2(\mathbb{R}^n)$ and $q(t,x) \in C(R^n)$ such that $f(t,x)$ has compact support and $q(t,x)$ is lower bounded.

(a) Define

$$V(t,x) = E^x\left[\left[\exp\left(-\int_0^t q(X(s))\,ds\right)\right] f(X(t))\right]. \tag{2.8}$$

Then

$$\frac{\partial V(t,x)}{\partial x} = \mathcal{A}V(t,x) - qV(t,x), t > 0\ , x \in \mathbb{R}^n \tag{2.9}$$

and

$$V(0,x) = f(x)\ x \in \mathbb{R}^n. \tag{2.10}$$

(b) Also if $\omega(t,x) \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ is a bounded function satisfying (2.9) and (2.10) then

$$\omega(t,x) = V(t,x) = E^x \left[\left[\exp\left(-\int_0^t q(X(s))\, ds \right) \right] f(X(t)) \right]. \tag{2.11}$$

2.3.10 Financial Market

Financial Market is the broad term describing any marketplace where buyers and sellers participate in the trade of assets such as equities, bonds, currencies and derivatives. Financial markets are typically defined by having transparent pricing, basic regulations on trading, costs and fees and market forces determining the prices of securities that trade. Some financial markets only allow participants that meet certain criteria, which can be based on factors like the amount of money held, the investor's geographical location, knowledge of the markets or the profession of the participant (Cvitanić and Zapatero, 2004).

In addition, Financial Market enables exchange of previously issued financial assets, financial markets facilitate borrowing and lending by facilitating the sale by newly issued financial assets. Examples of financial markets include the Dar es salaam Stock Exchange (resale of previously issued stock shares), the U.S. government bond market (resale of previously issued bonds), and the U.S. Treasury bills auction (sales of newly issued T-bills).

Mathematically, a financial market is an $\mathcal{F}_t^{(m)}-$ adapted, $(m+1)-$ dimensional process $X(t) = (X_0(t), X_1(t), X_2(t), \ldots\ldots., X_n(t))$ with $0 \leq t \leq T$ such that the first asset is a bond $dX_0(t) = \beta(t)X_0(t)dt \;\; ; X_0(0) = 1$ and the other $n-1$ assets are stocks.

2.3.11 Portfolio

The term portfolio refers to any collection of financial assets such as stocks, bonds, and cash. Portfolios may be held by individual investors and/or managed by financial professionals, hedge funds, banks and other financial institutions. It is a generally accepted principle that a portfolio is designed according to the investor's risk tolerance, time frame and investment objectives.

Mathematically, a portfolio in the market $X(t)_{t\in[0,T]}$ is an $(n+1)-$dimensional, $(t,\omega)-$ measurable and an $\mathcal{F}_t^{(m)}-$ adapted stochastic process which is given by:

$\theta(t,\omega) = \theta(t) = (\theta_0(t), \theta_1(t), \theta_2(t), ..., \theta_n(t))$ with $0 \leq t \leq T$.

2.3.12 Self Financing Trading Strategy

Given N assets with values $X_1(t), ..., X_N(t)$ at time t, a trading strategy is an $N-$ dimensional stochastic process $\theta_1(t), ..., \theta_N(t)$ that represents the allocations into the assets at time t. The time t value of the portfolio is $\Pi(t) = \sum_{i=1}^{N} \theta_i(t) X_i(t)$. The trading strategy is self financing if the change in the value of the portfolio is due only to changes in the value of the assets and not to inflows or outflows of funds.

This implies the strategy is self financing if:

$$d\Pi(t) = d\left(\sum_{i=1}^{N} \theta_i(t) X_i(t)\right) = \sum_{i=1}^{N} \theta_i(t) dX_i(t) \tag{2.12}$$

in other words, if:

$$\Pi(t) = \Pi(0) + \sum_{i=1}^{N} \int_0^N \theta_i(u) dX_i(u). \tag{2.13}$$

In the case of two assets the portfolio value is $\Pi(t) = \theta_1(t)X_1(t) + \theta_2(t)X_2(t)$ and the strategy (θ_1, θ_2) is self financing if $d\Pi(t) = \theta_1(t)dX_1(t) + \theta_2(t)dX_2(t)$ (Shreve 2004).

2.3.13 Arbitrage Opportunity

An arbitrage opportunity is a self financing trading strategy that produces the following properties on the portfolio value:

$$\begin{cases} \Pi(0) \leq 0. \\ P[\Pi(T) > 0] = 1. \end{cases} \tag{2.14}$$

This implies that the initial value of the portfolio (at time zero) is zero or negative, and the value of the portfolio at time T will be greater than zero with absolute certainty. This means that we start with a portfolio with zero value, or with debt (negative value). At some future time we have positive wealth, and since the strategy is self financing, no funds are required to produce this wealth. This is also known as “free lunch” although economists believe that arbitrage does not exist (Cvitanić and Zapatero, 2004).

2.3.14 Derivatives and Replication

The payoff $V(T)$ at time T of a derivative is a function of a risky asset. To rule out arbitrage we identify a self financing trading strategy that produces the same payoff as the derivative, so that $\Pi(T) = V(T)$. The trading strategy is then a replicating strategy and the portfolio is a replicating portfolio. If a replicating strategy exists the derivative is attainable, and if all derivatives are attainable the economy is complete.

In the absence of arbitrage the trading strategy produces a unique value for the value $V(T)$ of the derivative, otherwise an arbitrage opportunity would exist. Not only that, but also at every time t the value of the derivative, $V(t)$ must be equal to the value of the replicating strategy, $\Pi(t)$, so that $\Pi(t) = V(t)$. Otherwise an arbitrage opportunity exists (Cvitanić and Zapatero, 2004).

CHAPTER THREE

MODEL FORMULATION

3.1 Introduction

In this chapter, we formulate Black−Scholes partial differential equations (PDE) model after obtaining a market price $X(t)$ from a given mean reverting stochastic differential equation (SDE). Enough and relevant assumptions are made and the knowledge of martingale and It$\hat{o}$ formula are used.

3.2 Finding the market price

The underlying security (asset) price is given by the mean reverting geometric Brownian motion of the form

$$dX(t) = k(\rho - \ln X(t))X(t)dt + \sigma X(t)d\tilde{B}(t). \tag{3.1}$$

where k,ρ and σ are positive constants in which

- ⇒ k is a constant that governs the degree of mean reversion of the process. A high k implies a strong reversion of the process and vice versa.
- ⇒ ρ is the long run equilibrium level of an asset price also known as interest rate.
- ⇒ σ is an instantaneous standard deviation of the relative change also known as volatility.

$\tilde{B}$ is a Brownian motion with respect to the risk neutral measure Q given by the Girsanov theorem, that is on probability space $(\Omega, \mathcal{F}, Q)$. This assumption is very important in a sense that it differentiate this work from that of Shreve (2004).

From equation (3.1), $X(t)$ is an It$\hat{o}$ process which express our market price. To obtain this market price using equation (3.1) we use It$\hat{o}$ formula directly as follows:

$$\text{Let} \quad Y(t) = g(t, X(t)) = \ln X(t). \tag{3.2}$$

Such that $Y(t)$ is again an It$\hat{o}$ process. Thus by It$\hat{o}$ formula we have;

$$\begin{aligned} dY(t) = dg(t, X(t)) = d\ln X(t) = \frac{\partial g(t, X(t))}{\partial t}dt + \frac{\partial g(t, X(t))}{\partial X(t)}dX(t) \\ + \frac{1}{2}\frac{\partial^2 g(t, X(t))}{\partial X^2(t)}(dX(t))^2. \end{aligned} \tag{3.3}$$

From equations (2.4) and (3.1) above, since $(dX(t))^2 = \sigma^2 X^2(t)dt$ and using equation (3.2) we have;

$$\frac{\partial g(t, X(t))}{\partial t} = 0 \ , \quad \frac{\partial g(t, X(t))}{\partial X(t)} = \frac{1}{X(t)} \ \text{ and } \ \frac{\partial^2 g(t, X(t))}{\partial X^2(t)} = -\frac{1}{X^2(t)}. \tag{3.4}$$

Therefore equation (3.3) gives:

$$dg(t, X(t)) = \frac{1}{X(t)}dX(t) - \frac{\sigma^2}{2}dt. \tag{3.5}$$

Substituting equation (3.1) into equation (3.5) we have:

$$dg(t, X(t)) = \frac{1}{X(t)}\left[k(\rho - \ln X(t))X(t)dt + \sigma X(t)d\tilde{B}(t)\right] - \frac{\sigma^2}{2}dt. \tag{3.6}$$

which can be written as:

$$dg(t, X(t)) = k(\rho - \ln X(t))dt + \sigma d\tilde{B}(t) - \frac{\sigma^2}{2}dt. \tag{3.7}$$

Arrangement of terms gives:

$$dg(t, X(t)) + k\ln X(t)dt = \left(k\rho - \frac{\sigma^2}{2}\right)dt + \sigma d\tilde{B}(t). \tag{3.8}$$

Using equation (3.2), we have:

$$dY(t) + kY(t)dt = \left(k\rho - \frac{\sigma^2}{2}\right)dt + \sigma d\tilde{B}(t). \tag{3.9}$$

By the use of integrating factor

$$I.F = e^{\int k dt} = e^{kt},$$

in equation (3.9) one gets

$$e^{kt} dY(t) + e^{kt} kY(t)dt = e^{kt} \left(k\rho - \frac{\sigma^2}{2}\right) dt + e^{kt} \sigma d\tilde{B}(t). \tag{3.10}$$

Then equation (3.10) implies that

$$d\left[e^{kt} Y(t)\right] = e^{kt} \left(k\rho - \frac{\sigma^2}{2}\right) dt + e^{kt} \sigma d\tilde{B}(t). \tag{3.11}$$

Since we are after market price $X(t)$ at time t which is within $0 < t < T$, we take the integral of equation (3.11) from 0 to t as follows:

$$\int_0^t d\left[e^{ks} Y(s)\right] = \int_0^t e^{ks} \left(k\rho - \tfrac{\sigma^2}{2}\right) ds + \int_0^t e^{ks} \sigma d\tilde{B}(s)$$

$$\left[e^{ks} Y(s)\right]_0^t = \left[\left(k\rho - \tfrac{\sigma^2}{2}\right) \tfrac{e^{ks}}{k}\right]_0^t + \sigma \int_0^t e^{ks} d\tilde{B}(s)$$

$$e^{kt} Y(t) - Y(0) = \left[\left(k\rho - \tfrac{\sigma^2}{2}\right) \tfrac{e^{kt}}{k} - \left(k\rho - \tfrac{\sigma^2}{2}\right) \tfrac{1}{k}\right] + \sigma \int_0^t e^{ks} d\tilde{B}(s)$$

$$e^{kt} Y(t) - Y(0) = \left(\rho e^{kt} - \tfrac{\sigma^2 e^{kt}}{2k}\right) - \left(\rho - \tfrac{\sigma^2}{2k}\right) + \sigma \int_0^t e^{ks} d\tilde{B}(s)$$

$$e^{kt} Y(t) - Y(0) = e^{kt} \left(\rho - \tfrac{\sigma^2}{2k}\right) - \left(\rho - \tfrac{\sigma^2}{2k}\right) + \sigma \int_0^t e^{ks} d\tilde{B}(s)$$

$$e^{kt} Y(t) - Y(0) = \left(\rho - \tfrac{\sigma^2}{2k}\right) (e^{kt} - 1) + \sigma \int_0^t e^{ks} d\tilde{B}(s).$$

Multiply by e^{-kt} throughout to get

$$Y(t)-e^{-kt}Y(0)=\left(\rho-\tfrac{\sigma^2}{2k}\right)(1-e^{-kt})+\sigma e^{-kt}\textstyle\int_0^t e^{ks}d\tilde{B}(s).$$

Thus, we have:

$$Y(t)=e^{-kt}Y(0)+\left(\rho-\frac{\sigma^2}{2k}\right)(1-e^{-kt})+\sigma e^{-kt}\int_0^t e^{ks}d\tilde{B}(s). \tag{3.12}$$

Substituting equation (3.2) into equation (3.12) gives:

$$\ln X(t)=e^{-kt}\ln X(0)+\left(\rho-\frac{\sigma^2}{2k}\right)(1-e^{-kt})+\sigma e^{-kt}\int_0^t e^{ks}d\tilde{B}(s). \tag{3.13}$$

Therefore the market price $X(t)$ can be obtained from equation (3.13) as follows:

$$X(t)=\exp\left[e^{-kt}\ln X(0)+\left(\rho-\frac{\sigma^2}{2k}\right)(1-e^{-kt})+\sigma e^{-kt}\int_0^t e^{ks}d\tilde{B}(s)\right]. \tag{3.14}$$

3.3 Formulation of the Black−Scholes PDE model

3.3.1 Assumptions of the model

In order to formulate the Black−Scholes PDE model under the market whose price dynamics are due to mean reverting process, the following assumptions are taken into consideration.

(a) An asset has a “price” and of course it has as we have seen above, the price is given by equation (3.14).

(b) It is possible to set up replicating portfolio for the asset.

(c) Constant volatility.

The most significant assumption is that volatility, a measure of how much an asset can be expected to move in the near-term, is a constant over time.

(d) Interest rates constant and known.

The same like with the volatility, interest rates are also assumed to be constant in the model. The model uses the risk-free rate to represent this constant and known rate.

(e) A trading strategy is self financing it generates no dividends during the option's life that is for any time t in $0 < t < T$.

(f) No commissions and transaction costs.

The model assumes that there are no fees for buying and selling options and assets and no barriers to trading.

(g) Lognormally distributed returns.

The model assumes that returns on the underlying asset are normally distributed.

(h) Market has no arbitrage.

It is impossible to secure a risk free profit. Although there is arbitrage in certain market segments, these are not secure in the long run and relying on them violates that basic needs for our model to work.

(i) Efficient markets.

This assumption of the model suggests that people cannot consistently predict the direction of the market or an individual asset. The model assumes assets move in a manner referred to as a mean reverting. Mean reverting means that at any given moment in time, the price of the underlying asset can go up or down but there is probability almost sure of going back to the mean value. The price of an asset in the future is not predictable by using the current price, this is to say that price of the asset at time $t+1$ is independent from the price at time t that is, it is not Markovian.

(j) Liquidity of the market.

The model assumes that markets are perfectly liquid and it is possible to purchase or sell any amount of asset or their fractions at any given time.

3.3.2 Formulating the model

To formulate the Black−Scholes PDE model we assume the existence of two instruments:

(i) An underlying security $X(t)$ which evolves in accordance with the mean reverting geometric Brownian motion

$$dX(t) = k(\rho - \ln X(t))X(t)dt + \sigma X(t)d\tilde{B}(t). \tag{3.15}$$

(ii) The option price $u(t,x)$ at time t written on the underlying security $X(t)$, which is given by the value function

$$u(t,x) = E^{Q}\left[e^{-\rho(T-t)}F(\omega)|\mathcal{F}(t)\right] \tag{3.16}$$

where $F(\omega)$ is the payoff of a given barrier option.

We shall also use two facts as stated below:

(i) Itô integral is always martingale.

(ii) $e^{-\rho t}u(t,x)$ is a martingale with respect to Q.

By using iterated conditioning argument, equation (3.16) become:

$$e^{-\rho t}u(t,x) = E^{Q}\left[e^{-\rho T}F(\omega)|\mathcal{F}(t)\right]. \tag{3.17}$$

From equation (3.17), since $e^{-\rho t}u(t,x)$ is a martingale with respect to Q now we use Itô formula to formulate the Black−Scholes PDE model as follows.

Let $Z(t) = e^{-\rho t}u(t,x)$. Then by using Itô formula we have:

$$\begin{aligned} dZ(t) = -\rho e^{-\rho t}u(t,x)dt + e^{-\rho t}\frac{\partial u(t,x)}{\partial t}dt + e^{-\rho t}\frac{\partial u(t,x)}{\partial x}dX(t) \\ + \frac{1}{2}e^{-\rho t}\frac{\partial^2 u(t,x)}{\partial x^2}dX(t)dX(t). \end{aligned} \tag{3.18}$$

But $dX(t)dX(t) = \sigma^2 x^2 dt$. Then equation (3.18) become:

$$\begin{aligned} dZ(t) = -\rho e^{-\rho t}u(t,x)dt + e^{-\rho t}\frac{\partial u(t,x)}{\partial t}dt + e^{-\rho t}\frac{\partial u(t,x)}{\partial x}dX(t) \\ + \frac{1}{2}e^{-\rho t}\sigma^2 x^2\frac{\partial^2 u(t,x)}{\partial x^2}dt. \end{aligned} \tag{3.19}$$

Substituting equation (3.15) into equation (3.19) we get

$$\begin{aligned} dZ(t) = -\rho e^{-\rho t}u(t,x)dt + e^{-\rho t}\frac{\partial u(t,x)}{\partial t}dt + \frac{1}{2}e^{-\rho t}\sigma^2 x^2\frac{\partial^2 u(t,x)}{\partial x^2}dt \\ + e^{-\rho t}\frac{\partial u(t,x)}{\partial x}\left(k(\rho - \ln x)xdt + \sigma x d\tilde{B}(t)\right). \end{aligned} \tag{3.20}$$

Since $Z(t) = e^{-\rho t}u(t,x)$ then, equation (3.20) above can also be written as follows:

$$\begin{aligned} d\left(e^{-\rho t}u(t,x)\right) = e^{-\rho t}\sigma x\frac{\partial u(t,x)}{\partial x}d\tilde{B}(t) + e^{-\rho t}\bigg[-\rho u(t,x) + \frac{\partial u(t,x)}{\partial t} \\ + k(\rho - \ln x)x\frac{\partial u(t,x)}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 u(t,x)}{\partial x^2}\bigg]dt. \end{aligned} \tag{3.21}$$

Since $e^{-\rho t}u(t,x)$ is a martingale then dt term equals zero so that we remain with the Itô integral $d\left(e^{-\rho t}u(t,x)\right) = e^{-\rho t}\sigma x\frac{\partial u(t,x)}{\partial x}d\tilde{B}(t)$ which is also martingale.

But $e^{-\rho t} \neq 0$ for any values of ρ and t then it follows that:

$$-\rho u(t,x) + \frac{\partial u(t,x)}{\partial t} + k(\rho - \ln x)x\frac{\partial u(t,x)}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 u(t,x)}{\partial x^2} = 0. \tag{3.22}$$

Equation (3.22) above can be written as follows:

$$\frac{\partial u(t,x)}{\partial t} + k(\rho - \ln x)x\frac{\partial u(t,x)}{\partial x} + \frac{1}{2}\sigma^2 x^2\frac{\partial^2 u(t,x)}{\partial x^2} - \rho u(t,x) = 0. \tag{3.23}$$

Equation (3.23) above is the Black−Scholes PDE model for the value function given by $u(t,x) = E^Q\left[e^{-\rho(T-t)}F(\omega)|\mathcal{F}(t)\right]$ whose boundary conditions are $u(t,0) = 0$ and $u(t,L) = 0$ where L is the barrier level.

CHAPTER FOUR

MODEL ANALYSIS

4.1 Introduction

We begin this chapter by formulating a full boundary value problem by adding to the Black−Scholes PDE model (3.23) the boundary conditions for the chosen barrier option. We will then introduce the basic ideas and brief history of the Homotopy Analysis Method (HAM). Then the solution of the formed boundary value problem which is the price of a chosen barrier option will be obtained through the HAM. There after we shall analyse the price obtained by studying the effect of varying each parameter while other parameters remain constant.

4.2 Determination of the prices of the barrier options that satisfies the Black-Scholes PDE model

We have seen earlier that we have eight types of barrier options. We shall determine the price of up−and−out call barrier option as an example. The remaining prices of the barrier options can be determined in a similar way.

4.2.1 Boundary value problem for the up−and−out European call

Using (3.23) and up−and−out European call then upon adding the boundary conditions, a full boundary value problem for this case become:

$$\frac{\partial u(t,x)}{\partial t} + k(\rho - \ln x)x\frac{\partial u(t,x)}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u(t,x)}{\partial x^2} - \rho u(t,x) = 0 \tag{4.1}$$

$$u(t,0) = 0, \qquad 0 \le t \le T \tag{4.2}$$

$$u(t,L) = 0, \qquad 0 \le t < T \tag{4.3}$$

$$u(T,x) = (x-K)^+, \qquad 0 \le x \le L \tag{4.4}$$

where L is the barrier and K is the strike price.

The remaining task is to obtain $u(t,x)$ which represent the required price from the above boundary value problem, to achieve this aim in the next subsections we explain the basic ideas and brief history of the HAM, this is the method that we will use.

4.2.2 Basic ideas and brief history of the HAM

The homotopy analysis method (HAM) is the method that can be used to solve nonlinear differential equations. The homotopy analysis method employs the concept of the homotopy from topology to flexibly generate a convergent series solution for nonlinear systems. This is especially enabled by utilizing a so-called homotopy-Mclaurin series to deal with the nonlinearities in the system (Liao, 2012).

In most cases it is difficult to solve nonlinear problems, especially analytically. In general, there are two standards for a satisfactory method of nonlinear problems:

(a) It can always provide approximations efficiently.

(b) It can ensure that approximations are accurate enough for all physical parameters.

Using above two standards as criterion, let us compare different techniques for nonlinear problems.

Perturbation techniques comprises of mathematical methods for finding an approximate solution to a problem which cannot be solved exactly, by starting from the exact solution of a related problem they are based on the existence of small/large parameters, the so-called perturbation quantity. Unfortunately, many nonlinear problems in science and engineering do not contain such kind of perturbation quantities at all. Some nonperturbative techniques, such as the artificial small parameter method, the delta expansion method and the Adomian's decomposition method, have been developed (Wikipedia, 2013).

Different from perturbation techniques, these nonperturbative methods are independent upon small parameters. However, both of the perturbation techniques and the nonperturbative methods themselves can not provide us with a simple way to adjust or control the convergence region and rate of given approximate series.

Liao (1992) proposed a powerful method for nonlinear problems, namely the homotopy analysis method (HAM). Different from all reported perturbation and nonperturbative techniques mentioned above, the homotopy analysis method itself provides us with a convenient way to control and adjust the convergence region and rate of approximation series, when necessary. Briefly speaking, the homotopy analysis method has the following advantages:

First, it is a series expansion method but it is entirely independent of small physical parameters. Thus, it is applicable for not only weakly but also strongly nonlinear problems, going beyond some of the limitations well known in perturbation methods.

Secondly, the HAM is an unified method for the Lyapunov artificial small parameter method, the delta expansion method, the Adomian's decomposition method, and the homotopy perturbation method. The greater generality of the method often allows for strong convergence of the solution over larger spacial and parameter domains.

Thirdly, the HAM gives excellent flexibility in the expression of the solution and how the solution is explicitly obtained. It provides a simple way to ensure the convergence of the solution, freedom to choose the basis functions of the desired solution, and flexibility in determining the linear operator of the homotopy.

Fourthly, in conjunction with symbolic computation, the HAM can be combined with many other standard mathematical methods such as numerical methods, series expansion methods, integral transform methods, and so forth (Wikipedia, 2013).

For better description of the basic ideas of the homotopy analysis method, let us consider a given nonlinear differential equation given by equation (4.5) below:

$$\mathcal{N}[u(x)] = 0, x \in \Omega \tag{4.5}$$

where $\mathcal{N}$ is a nonlinear operator and $u(x)$ is a unknown function. Liao (1992) constructed a one-parameter family of equations in the embedding parameter $q \in [0,1]$, called the zeroth-order deformation equation:

$$(1-q)\mathcal{L}[\phi(x;q) - u_0(x)] + q\mathcal{N}[\phi(x;q)] = 0, x \in \Omega, q \in [0,1] \tag{4.6}$$

where $\mathcal{L}$ is an auxiliary linear operator (in this study $\frac{\partial}{\partial t}$ will be used) and $u_0(x)$ is an initial guess. In theory, the concept of homotopy in topology provides us much larger freedom to choose both of the auxiliary linear operator and the initial guess than the traditional non-perturbation methods mentioned above, as pointed out by Liao.

At $q = 0$ and $q = 1$, we have $\phi(x;0) = u_0(x)$ and $\phi(x;1) = u(x)$, respectively. So, as the embedding parameter $q \in [0,1]$ increases from 0 to 1, the solution $\phi(x;q)$ of the zeroth-order deformation equations varies (or deforms) from the initial guess $u_0(x)$ to the exact solution $u(x)$ of the original nonlinear differential equation $\mathcal{N}[u(x)] = 0$. According to Liao (1992) such kind of continuous variation is called deformation in topology, and this is the reason why they call equation (4.6) the zeroth-order deformation equation. Since $\phi(x;q)$ is also dependent upon the embedding parameter $q \in [0,1]$, we can expand it into the Maclaurin series with respect to q as follows:

$$\phi(x;q) = u_0(x) + \sum_{n=1}^{+\infty} u_n(x) q^n. \tag{4.7}$$

Equation (4.7) above is known as the homotopy-Maclaurin series. Note that we have extremely large freedom to choose the auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(x)$. Assuming that,the auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(x)$ are so properly chosen such that the above homotopy-Maclaurin series converges at $q = 1$, we have the so-called homotopy-series solution given by:

$$u(x) = u_0(x) + \sum_{n=1}^{+\infty} u_n(x) \tag{4.8}$$

which satisfies the original equation (4.5).

Therefore with HAM we do not need linearisation and the solution is calculated in the form of a convergent power series which resolve non-linearity of the problem automatically as a result we obtain an approximated solution to a given non-linear problem.

4.2.3 Computation of the price of the up−and−out European call

By using the Homotopy Analysis Method (HAM) explained above, the boundary value problem obtained earlier can be solved as shown below and the result obtained will be the approximation of the needed price of the up−and−out European call.

Consider equation (4.1) above:

$$\frac{\partial u(t,x)}{\partial t}+k(\rho-\ln x)x\frac{\partial u(t,x)}{\partial x}+\frac{1}{2}\sigma^2x^2\frac{\partial^2u(t,x)}{\partial x^2}-\rho u(t,x)=0 \tag{4.9}$$

which is the Black-Scholes PDE model for the value function:

$u(t,x)=E^Q\left[e^{-\rho(T-t)}F(\omega)|\mathcal{F}(t)\right]$ whose boundary conditions are $u(t,0)=0$, $u(t,L)=0$ and $u(T,x)=(x-K)^+$ where L is the barrier level.

Upon choosing the auxiliary linear operator $\mathcal{L}$ and the initial guess $u_0(x)$ when we apply the HAM we have the following;

$$(1-q)\left[\frac{\partial v}{\partial t}-\frac{\partial u_0(x)}{\partial t}\right]+q\left[\frac{\partial v}{\partial t}+\frac{\partial v}{\partial x}xk(\rho-\ln x)+\frac{1}{2}\sigma^2x^2\frac{\partial^2v}{\partial x^2}-\rho v\right]=0,\quad q\in[0,1]. \tag{4.10}$$

Let the solution be of the form:

$$v=v_0+qv_1+q^2v_2+q^3v_3+... \tag{4.11}$$

then

$$\frac{\partial v}{\partial t}=\frac{\partial v_0}{\partial t}+q\frac{\partial v_1}{\partial t}+q^2\frac{\partial v_2}{\partial t}+q^3\frac{\partial v_3}{\partial t}+... \tag{4.12}$$

$$\frac{\partial v}{\partial x}=\frac{\partial v_0}{\partial x}+q\frac{\partial v_1}{\partial x}+q^2\frac{\partial v_2}{\partial x}+q^3\frac{\partial v_3}{\partial x}+... \tag{4.13}$$

$$\frac{\partial^2 v}{\partial x^2}=\frac{\partial^2 v_0}{\partial x^2}+q\frac{\partial^2 v_1}{\partial x^2}+q^2\frac{\partial^2 v_2}{\partial x^2}+q^3\frac{\partial^2 v_3}{\partial x^2}+... \tag{4.14}$$

Substituting equations (4.12), (4.13)and (4.14) into equation (4.10), we have the following:

$$(1-q)\Big[\frac{\partial v_0}{\partial t}+q\frac{\partial v_1}{\partial t}+q^2\frac{\partial v_2}{\partial t}+q^3\frac{\partial v_3}{\partial t}-\frac{\partial u_0(x)}{\partial t}\Big]+q\Big[\frac{\partial v_0}{\partial t}+q\frac{\partial v_1}{\partial t}+q^2\frac{\partial v_2}{\partial t}+ q^3\frac{\partial v_3}{\partial t}+xk(\rho-\ln x)\Big(\frac{\partial v_0}{\partial x}+q\frac{\partial v_1}{\partial x}+q^2\frac{\partial v_2}{\partial x}+q^3\frac{\partial v_3}{\partial x}\Big)+\frac{1}{2}\sigma^2x^2\Big(\frac{\partial^2 v_0}{\partial x^2}+q\frac{\partial^2 v_1}{\partial x^2}+ q^2\frac{\partial^2 v_2}{\partial x^2}+q^3\frac{\partial^2 v_3}{\partial x^2}\Big)-\rho\Big(v_0+qv_1+q^2v_2+q^3v_3\Big)\Big]=0. \tag{4.15}$$

Equating the terms with the same powers of q, we get

$$q^0:\frac{\partial v_0}{\partial t}-\frac{\partial u_0(x)}{\partial t}=0.$$

$$q^1:\frac{\partial v_1}{\partial t}+\frac{\partial u_0(x)}{\partial t}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 v_0}{\partial x^2}+xk(\rho-\ln x)\frac{\partial v_0}{\partial x}-\rho v_0=0.$$

$$q^2:\frac{\partial v_2}{\partial t}+xk(\rho-\ln x)\frac{\partial v_1}{\partial x}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 v_1}{\partial x^2}-\rho v_1=0.$$

$$q^3:\frac{\partial v_3}{\partial t}+xk(\rho-\ln x)\frac{\partial v_2}{\partial x}+\frac{1}{2}\sigma^2x^2\frac{\partial^2 v_2}{\partial x^2}-\rho v_2=0.$$

Suppose the initial guess is: $u_0(x)=u_0=[x(x-L)]^{(T-t)}(x-K)^+$ where $x>K$.

This implies that:

$$\frac{\partial u_0}{\partial t}=-(x-K)^+[x(x-L)]^{(T-t)}\ln[x(x-L)].$$

Substituting these equations into the equations obtained from the powers of q we get

For q^0 :

$$\frac{\partial v_0}{\partial t}-\frac{\partial u_0(x)}{\partial t}=0.$$

But $\frac{\partial u_0}{\partial t}=-(x-K)^+[x(x-L)]^{(T-t)}\ln[x(x-L)]$.

Then we have:

$$\frac{\partial v_0}{\partial t} = -(x-K)^+[x(x-L)]^{(T-t)}\ln[x(x-L)].$$

Integrate throughout with respect to t:

$$\int \partial v_0 = -\int (x-K)^+[x(x-L)]^{(T-t)}\ln[x(x-L)]\partial t.$$

Thus,

$$v_0 = [x(x-L)]^{(T-t)}(x-K)^+. \tag{4.16}$$

From equation (4.15) we have:

$$\frac{\partial v_0}{\partial x} = (x-K)^+(T-t)(2x-L)[x(x-L)]^{(T-t-1)} + [x(x-L)]^{(T-t)}.$$

$$\begin{aligned}\frac{\partial^2 v_0}{\partial x^2} =& (x-K)^+(2x-L)(T-t)(T-t-1)(2x-L)[x(x-L)]^{(T-t-2)} + [2(x-K)^+ \\ &+ (2x-L)](T-t)[x(x-L)]^{(T-t-1)} + (T-t)(2x-L)[x(x-L)]^{(T-t-1)}.\end{aligned}$$

For q^1 :

$$\frac{\partial v_1}{\partial t} + \frac{\partial u_0(x)}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v_0}{\partial x^2} + xk(\rho - \ln x)\frac{\partial v_0}{\partial x} - \rho v_0 = 0.$$

Then we have:

$$\frac{\partial v_1}{\partial t} - (x-K)^+[x(x-L)]^{(T-t)}\ln[x(x-L)] + \frac{1}{2}\sigma^2 x^2\Big[[2(x-K)^+ + (2x-L)](T-t)[x(x-L)]^{(T-t-1)} + (x-K)^+(2x-L)(T-t)(T-t-1)(2x-L)[x(x-L)]^{(T-t-2)} + (T-t)(2x-L)[x(x-L)]^{(T-t-1)}\Big] + xk(\rho - \ln x)\Big[(x-K)^+(T-t)(2x-L)[x(x-L)]^{(T-t-1)} + [x(x-L)]^{(T-t)}\Big] - \rho\Big[[x(x-L)]^{(T-t)}(x-K)^+\Big] = 0.$$

Integrate throughout with respect to t:

$$\int \partial v_1 = \int \bigg[(x-K)^+[x(x-L)]^{(T-t)} \ln[x(x-L)] - \frac{1}{2}\sigma^2 x^2 \Big[[2(x-K)^+ + (2x-L)](T-t)[x(x-L)]^{(T-t-1)} + (x-K)^+(2x-L)(T-t)(T-t-1)(2x-L)[x(x-L)]^{(T-t-2)} + (T-t)(2x-L)[x(x-L)]^{(T-t-1)} \Big] - xk(\rho - \ln x)\Big[[x(x-L)]^{(T-t)} + (x-K)^+(T-t)(2x-L)[x(x-L)]^{(T-t-1)} \Big] + \rho \Big[[x(x-L)]^{(T-t)}(x-K)^+ \Big] \bigg] \partial t.$$

Yields

$$\begin{aligned} v_1 = & - [x(x-L)]^{(T-t)}(x-K)^+ - \frac{1}{2}\sigma^2 x^2 \bigg[[2(x-K)^+ + (2x-L)][- \\ & \frac{(T-t)[x(x-L)]^{(T-t-1)}}{\ln(x(x-L))} + \frac{[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2}] + (x-K)^+(2x-L)^2 \\ & \Big[[- \frac{(T-t)[x(x-L)]^{(T-t-2)}}{\ln(x(x-L))} + \frac{[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2}](T-t-1) - \\ & \frac{1}{\ln(x(x-L))}[- \frac{(T-t)[x(x-L)]^{(T-t-2)}}{\ln(x(x-L))} + \frac{[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2}] - \\ & \frac{[x(x-L)]^{(T-t-2)}}{[\ln(x(x-L))]^3}\Big] + \frac{(2x-L)[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2} - \\ & \frac{(2x-L)(T-t)[x(x-L)]^{(T-t-1)}}{\ln(x(x-L))}\bigg] - xk(\rho - \ln x)\bigg[(x-K)^+(2x-L)[- \\ & \frac{(T-t)[x(x-L)]^{(T-t-1)}}{\ln(x(x-L))} + \frac{[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2}] - \frac{[x(x-L)]^{(T-t)}}{\ln(x(x-L))}\bigg] \\ & + \frac{\rho(x-K)^+[x(x-L)]^{(T-t)}}{\ln(x(x-L))}. \end{aligned}$$

Since the results for $v's$ becomes too long, we will use these two obtained values.

Substituting the values of v_0, v_1, v_2 and v_3 into equation (4.11) we have:

$$v = v_0 + qv_1 + q^2 v_2 + q^3 v_3 + \ldots.$$

Therefore

$$\begin{aligned} v = & [x(x-L)]^{(T-t)}(x-K)^{+} + q\Bigg[-[x(x-L)]^{(T-t)}(x-K)^{+} - \frac{1}{2}\sigma^2 x^2\Bigg[[2(x-K)^{+} + \\ & (2x-L)][-\frac{(T-t)[x(x-L)]^{(T-t-1)}}{\ln(x(x-L))} + \frac{[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2}] + (x-K)^{+}(2x-L)^2 \\ & \Bigg[[-\frac{(T-t)[x(x-L)]^{(T-t-2)}}{\ln(x(x-L))} + \frac{[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2}](T-t-1) - \frac{1}{\ln(x(x-L))}[\\ & -\frac{(T-t)[x(x-L)]^{(T-t-2)}}{\ln(x(x-L))} + \frac{[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2}] - \frac{[x(x-L)]^{(T-t-2)}}{[\ln(x(x-L))]^3}\Bigg] \\ & + \frac{(2x-L)[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2} - \frac{(2x-L)(T-t)[x(x-L)]^{(T-t-1)}}{\ln(x(x-L))}\Bigg] - \\ & xk(\rho - \ln x)\Bigg[(x-K)^{+}(2x-L)[-\frac{[x(x-L)]^{(T-t)}}{\ln(x(x-L))} - \frac{(T-t)[x(x-L)]^{(T-t-1)}}{\ln(x(x-L))} + \\ & \frac{[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2}]\Bigg] + \frac{\rho(x-K)^{+}[x(x-L)]^{(T-t)}}{\ln(x(x-L))}\Bigg] + \dots \end{aligned}$$

But

$$u = \lim_{q\to 1} v = v_0 + v_1 + v_2 + v_3.$$

Thus from above, we have:

$$\begin{aligned} u = & u(t,X(t)) = u(t,x) = -\frac{1}{2}\sigma^2 x^2\Bigg[[2(x-K)^{+} + (2x-L)][\frac{[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2}] - \\ & \frac{(T-t)[x(x-L)]^{(T-t-1)}}{\ln(x(x-L))} + (x-K)^{+}(2x-L)^2\Bigg[[-\frac{(T-t)[x(x-L)]^{(T-t-2)}}{\ln(x(x-L))} + \\ & \frac{[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2}](T-t-1) - \frac{1}{\ln(x(x-L))}[-\frac{(T-t)[x(x-L)]^{(T-t-2)}}{\ln(x(x-L))} + \\ & \frac{[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2}] - \frac{[x(x-L)]^{(T-t-2)}}{[\ln(x(x-L))]^3}\Bigg] + \frac{(2x-L)[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2} - \\ & \frac{(2x-L)(T-t)[x(x-L)]^{(T-t-1)}}{\ln(x(x-L))}\Bigg] - xk(\rho - \ln x)\Bigg[(x-K)^{+}(2x-L)[- \\ & \frac{(T-t)[x(x-L)]^{(T-t-1)}}{\ln(x(x-L))} + \frac{[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2}] - \frac{[x(x-L)]^{(T-t)}}{\ln(x(x-L))}\Bigg] + \\ & \frac{\rho(x-K)^{+}[x(x-L)]^{(T-t)}}{\ln(x(x-L))}. \end{aligned}$$

Therefore the price of the up−and−out European call is:

$$\begin{aligned}
u(t,x) = & -\frac{1}{2}\sigma^2 x^2 \Bigg[[2(x-K)^+ + (2x-L)][\frac{[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2}] \\
& -\frac{(T-t)[x(x-L)]^{(T-t-1)}}{\ln(x(x-L))} + (x-K)^+(2x-L)^2 \Bigg[[-\frac{(T-t)[x(x-L)]^{(T-t-2)}}{\ln(x(x-L))} + \\
& \frac{[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2}](T-t-1) - \frac{1}{\ln(x(x-L))}[-\frac{(T-t)[x(x-L)]^{(T-t-2)}}{\ln(x(x-L))} + \\
& \frac{[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2}] - \frac{[x(x-L)]^{(T-t-2)}}{[\ln(x(x-L))]^3} \Bigg] + \frac{(2x-L)[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2} - \\
& \frac{(2x-L)(T-t)[x(x-L)]^{(T-t-1)}}{\ln(x(x-L))} \Bigg] - xk(\rho - \ln x) \Bigg[(x-K)^+(2x-L)[- \\
& \frac{(T-t)[x(x-L)]^{(T-t-1)}}{\ln(x(x-L))} + \frac{[x(x-L)]^{(T-t-1)}}{[\ln(x(x-L))]^2}] - \frac{[x(x-L)]^{(T-t)}}{\ln(x(x-L))} \Bigg] + \\
& \frac{\rho(x-K)^+[x(x-L)]^{(T-t)}}{\ln(x(x-L))}. \qquad (4.17)
\end{aligned}$$

where $x = X(t)$ is the market/asset price which is a solution of the mean reverting geometric Brownian motion $dX(t) = k(\rho - \ln X(t))X(t)dt + \sigma X(t)d\tilde{B}(t)$.

The formula for the price of the up−and−out European call is derived under the assumption that the initial price of the up−and−out European call exist and it obeys the boundary conditions (4.2),(4.3) and (4.4) above.

We note that $u(t,0) = 0$ because It$\hat{o}$ integral starting at zero stays at zero and hence the call expires out of the money, this is the boundary condition (4.2) which can also be shown upon taking a limit of $u(t,x)$ as x approaches 0, although to achieve this, one needs to remember that:

$$\lim_{x\to 0} x\ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}}$$

by L'Hopital's rule we have:

$$\lim_{x\to 0} x\ln x = \lim_{x\to 0}\frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0}\frac{-x^2}{x} = \lim_{x\to 0} -x = 0$$

also he/she needs to remember that, by L'Hopital's rule we have:

$$\lim_{x\to 0}\frac{x}{\ln x} = \lim_{x\to 0}\frac{1}{\frac{1}{x}} = \lim_{x\to 0} x = 0.$$

For $0 \leq t \leq T$ and $x > L$, we have $u(t,x) = 0$ because the option knocks out when the asset price exceeds the barrier L. Indeed, if the asset price reaches the barrier before expiration, then it will immediately exceed the barrier almost surely, and so $u(t,L) = 0$ for $0 \leq t < T$, however $u(T,L) = L-K$, this is the boundary condition (4.3) which can also be shown upon taking a limit of $u(t,x)$ as x approaches L, in this case a reader also needs to apply L'Hopital's rule as explained above.

Finally, if the option does not knock out prior to expiration, then its payoff is that of a European call , that is , $u(T,x) = (x-K)^+$. This is the boundary condition (4.4) which can also be shown upon substituting $t = T$ in equation (4.17).

Therefore, in summary, $u(t,x)$ satisfies the boundary conditions (4.2), (4.3) and (4.4) in the boundary value problem as explained above.

4.3 Analysis of the parameters on the price of the barrier option

In this subsection an analysis on the parameters of the price of the barrier option obtained above is carried out to study the behaviour of the price when we vary one parameter keeping the other parameters constant. The process is performed for three parameters namely k , σ and ρ as a result. Plots will be drawn for each case and a conclusion will be extracted from those plots.

Since k , σ and ρ are the degree of mean reversion, volatility of the market and interest rate respectively. In order to analyse the behaviour of the price as a given parameter value increases, in this study we shall use values ranging from zero to fifteen some of them were used by other researchers while the remaining values were chosen randomly to check the behaviour of the price as value of a given parameter increases.

For k we have used 0.5, 1.0, 1.5, 2, 5 and 15. In this case 0.5, 1.0 and 2 were used by Hui and Lo (2006).

For ρ we have used 0.01, 0.05, 0.20, 2, 3 and 4. In this case 0.01 and 0.05 were used by Wong and Lau (2007). Also 0.20 was used by Wang, Fu and Marcus (2009).

For σ we have used 0.20, 0.25, 0.50, 2, 5 and 15. In this case 0.20, 0.25 and 0.50 were used by Fusari, Adesi and Theal (2010).

4.3.1 The price of the barrier option for different values of k with constant values of ρ and σ

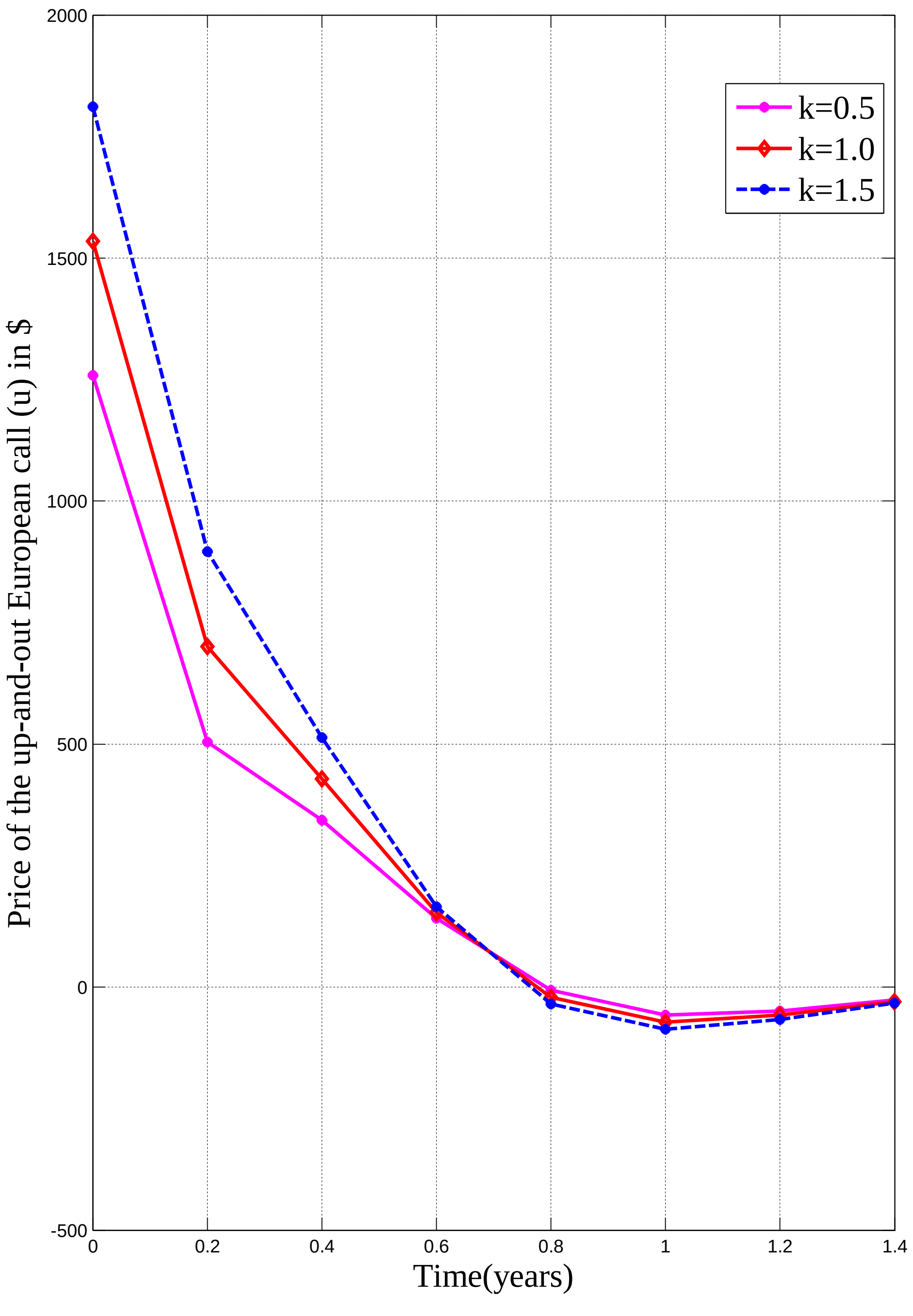

Figure 4.1: Price of the up−and−out European call for different values of k

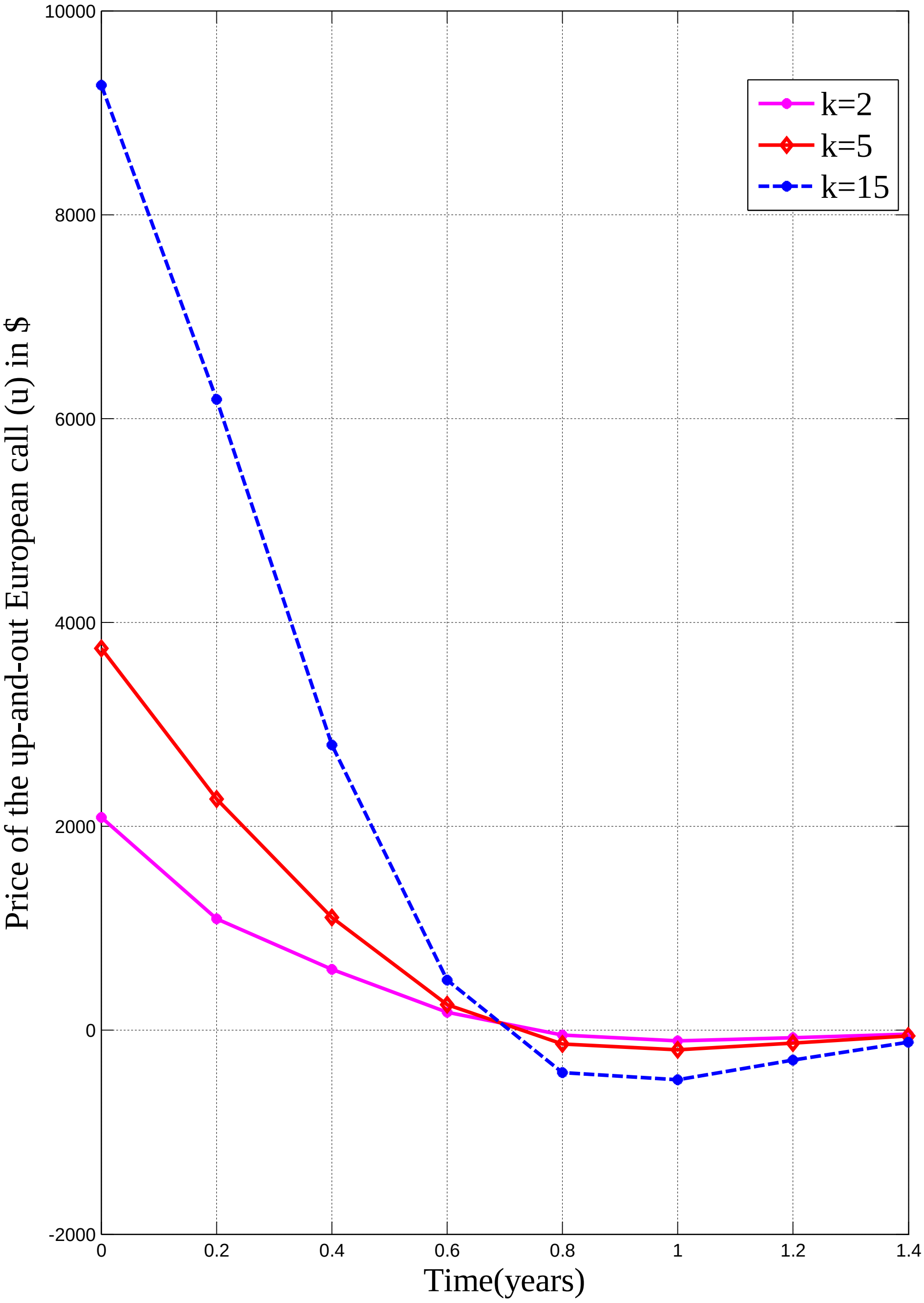

Figure 4.2: Price of the up−and−out European call for different values of *k*

Figures 4.1 and 4.2 above show that when k increases keeping other parameters constant the price also increases, as k increases rapidly the price also increase rapidly this implies that as a degree of mean reversion increases rapidly the price also increase rapidly. In other words it implies that a strong reversion leads to higher price and vice versa. Also the figure suggests that owner should sell the option very early within the first seven months due to the fact that the price tends to decrease as time increases.

4.3.2 The price of the barrier option for different values of ρ with constant values of σ and k

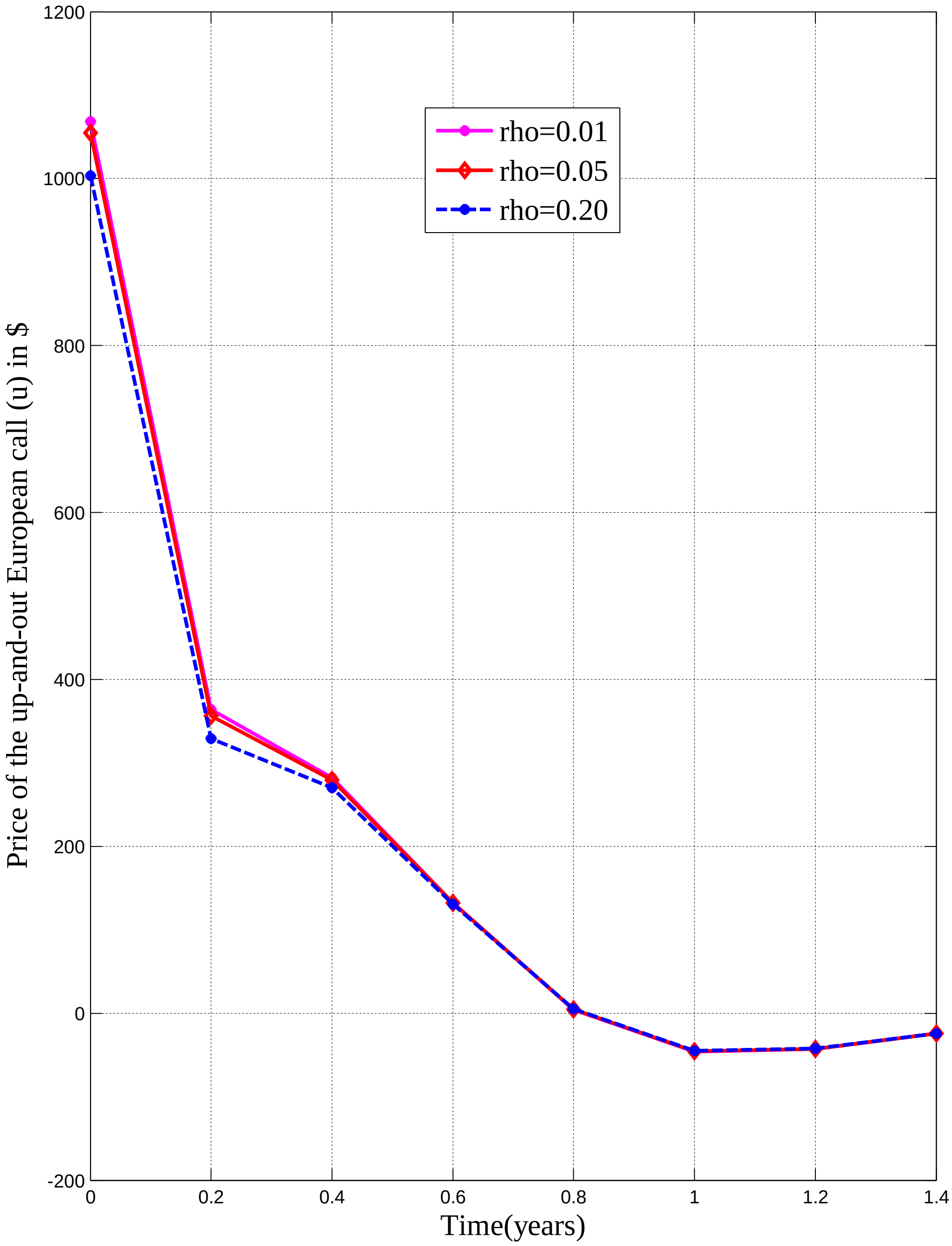

Figure 4.3: Price of the up−and−out European call for different values of ρ

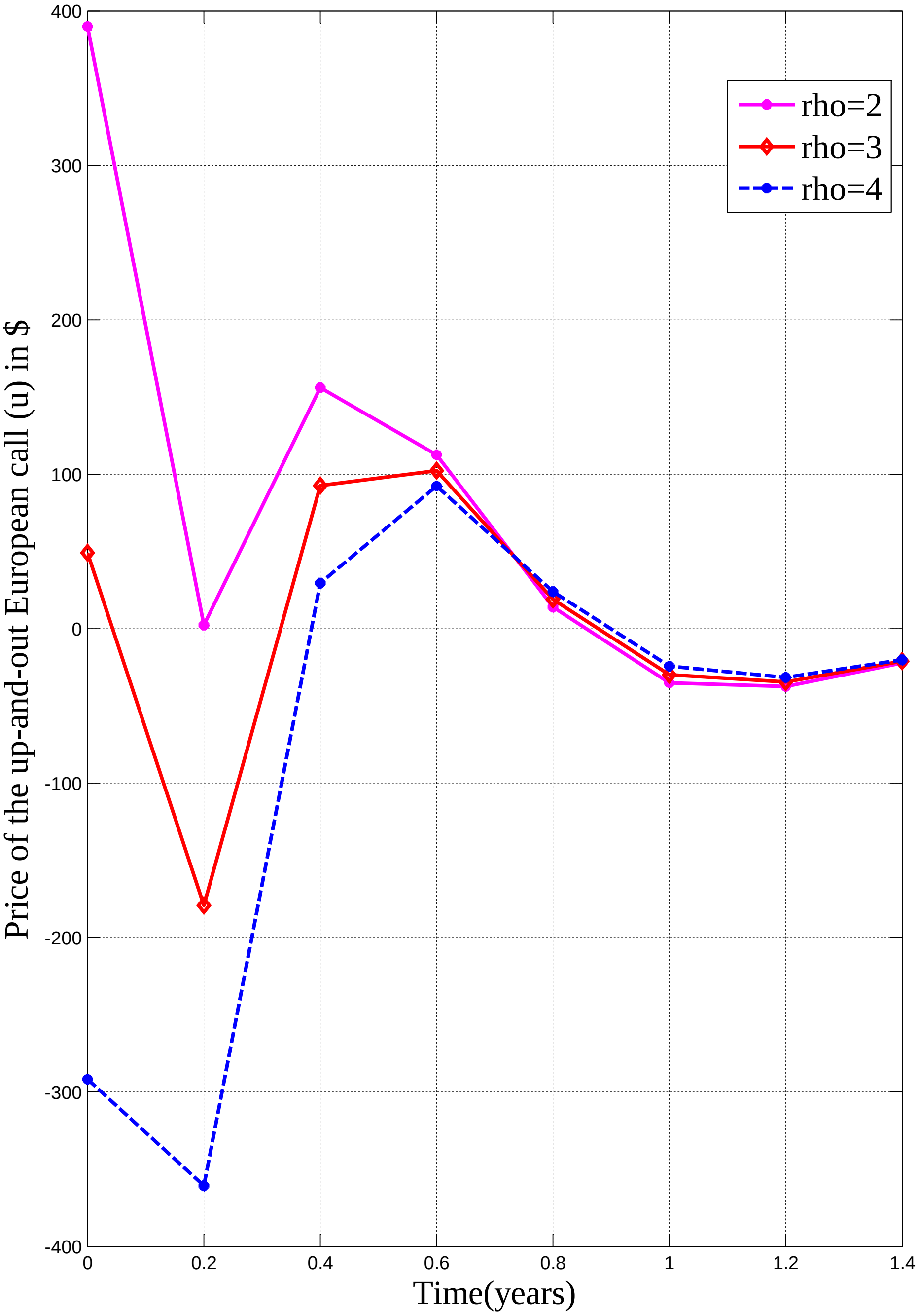

Figure 4.4: Price of the up−and−out European call for different values of ρ

Figures 4.3 and 4.4 above show that when ρ increases keeping other parameters constant the price decreases this implies that as interest rate increases the price decreases. In other words it implies that a higher interest rate leads to a lower price and also it can be observed that an interest rate, ρ should not be too large since it will lead to negative price which is meaningless. In addition when the interest rate is between 1 and 3 then the owner of the option should sell the option very early within the first two months.

4.3.3 The price of the barrier option for different values of σ with constant values of k and ρ

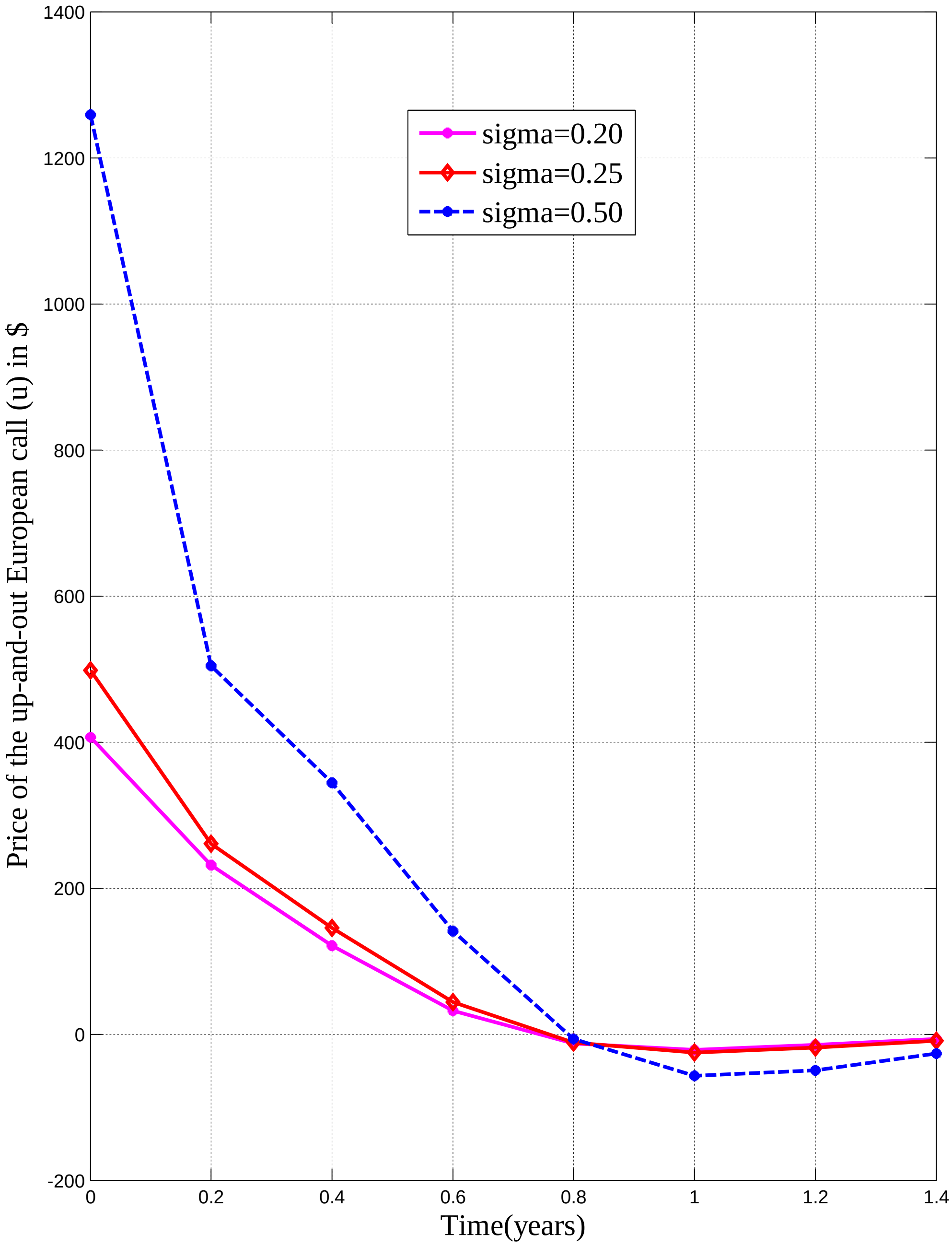

Figure 4.5: Price of the up−and−out European call for different values of σ

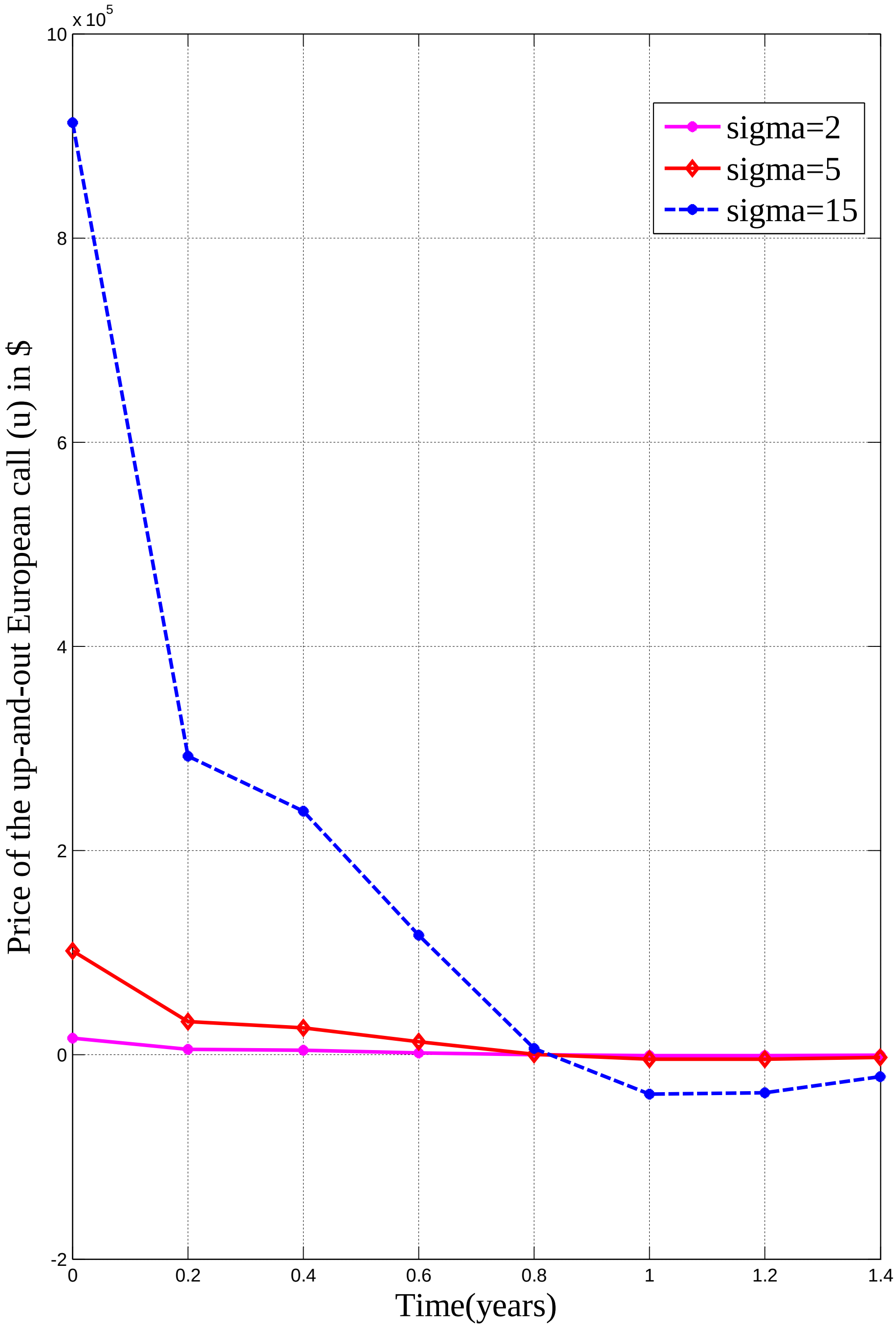

Figure 4.6: Price of the up−and−out European call for different values of σ

Figures 4.5 and 4.6 above show that when σ increases keeping other parameters constant the price also increases this implies that as volatility of the market increases the price also increase. When volatility of the market become too large the price increases rapidly going to infinity, thus the figure suggests that owner should exercise the option within the first nine months or when the volatility of the market become large.

CHAPTER FIVE

SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Summary

This dissertation is about studying the problem of pricing of barrier options when the dynamics of the prices are driven by the mean reverting process. Chapter one starts by giving a general introduction and defining terms related to barrier options and option at large. Statement of the problem is then stated followed by research objectives, significance of the study, research hypotheses and lastly methodology of the study. Chapter two contains relevant literature and theoretical background about. The first specific objective of this study is achieved in chapter three where we formulate Black−Scholes partial differential equations (PDE) model after obtaining a market price $X(t)$ from a given mean reverting stochastic differential equation (SDE).

Chapter four starts by formulating a full boundary value problem by adding to the Black−Scholes PDE model (obtained in chapter three) the boundary conditions for the chosen barrier option, there after we introduced the basic ideas and brief history of the Homotopy Analysis Method (HAM) then the second specific objective is achieved through finding a solution of the formed boundary value problem which is the price of a chosen barrier option by using HAM and there after an analysis was conducted on the price obtained to study the effect of varying each parameter while other parameters remain constant, here the third specific objective is achieved . Finally chapter five gives the summary, conclusion and recommendations for the future works.

5.2 Conclusion

Main problem in this dissertation was to determine prices of barrier options when the dynamics of the prices are driven by the mean reverting process, model analysis from the study indicate that the Homotopy Analysis Method (HAM) can be used to determine the approximated prices of barrier options when the dynamics of the prices are driven by the mean reverting process.

Analysis of the result through variation of the parameter in the price obtained shows that the price tends to increase with the increase of the parameter for the case of volatility and degree of mean reversion while for interest rate the price decreases when interest rate increases. It was observed that early exercise is better than late exercise to owner of the option since the price tends to decrease as time increases also to minimize risk the owner of the option should exercise the option when the volatility of the market become large.

5.3 Recommendations

Developing a technique for pricing barrier options when the dynamics of the prices are driven by the mean reverting process is among the researchable and interesting areas in financial mathematics, and it has become increasingly important in the financial markets due to the fact that barrier options are cheap and mean reversion is widely associated with commodities of all sorts such as copper, oil, money. Practically, the results of this study can be used in financial markets to price up$-$and$-$out European call option. In future, recommended areas of research which relate to this study are:

(i) Finding the price by using direct integration after obtaining a reflection principle which is useful in determining the joint distribution of the It$\hat{o}$ integral.

(ii) Finding the price which is a closed form solution by using Laplace transform.

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APPENDIX

MATLAB CODES USED IN ANALYSIS OF THE PARAMETERS OF THE PRICE OF THE BARRIER OPTION

Matlab Codes for plots which shows the price of the up−and−out European call for different values of *k*

```
clear
close all
x=15;
K=10;
L=20;
T=1.4;
k=0.5;
sigma=0.5;
rho=0.1;
t=0:0.2:1.4;
u=(((-sigma^2)*(x^2))/2).*[[(2.*(x-K))+(2.*x-L)].*[-((T-t).
*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L)))+(([x.*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))]+[(x-K).*(2.*x-L)^2].*[[(-((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))+ (([x*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))].*(T-t-1)-((1)/(log(x.*(x-L)))).
*[((([x.*(x-L)].^(T-t-1))/(log((x.*(x-L))^2)))-(((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))]-(([x.*(x-L)].
^(T-t-2))/(log((x.*(x-L))^3)))]+(((2.*x-L).*[x.*(x-L)].
^(T-t-1))/(log((x.*(x-L))^2)))-(((2.*x-L).*(T-t).*[x.*(x-L)].
^(T-t-1))/(log(x.*(x-L))))]-x.*k.*(rho-log(x)).*[(x-K).
*(2.*x-L).*[-(((T-t).*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L))))+
(([x.*(x-L)].^(T-t-1))/((log(x.*(x-L)))^2))]-(([x.*(x-L)].
^(T-t))/(log(x.*(x-L))))]+(((rho.*(x-K)).*[x.*(x-L)].^(T-t))/
(log(x.*(x-L))));
plot(t,u,'m*-','LineWidth',2)
grid on;
xlabel('Time(years)')
ylabel('Price of the up-and-out European call (u)')
hold on
k=1;
u=(((-sigma^2)*(x^2))/2).*[[(2.*(x-K))+(2.*x-L)].*[-((T-t).
```

```
*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L)))+(([x.*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))]+[(x-K).*(2.*x-L)^2].*[[(-((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))+ (([x*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))].*(T-t-1)-((1)/(log(x.*(x-L)))).
*[(([x.*(x-L)].^(T-t-1))/(log((x.*(x-L))^2)))-(((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))]-(([x.*(x-L)].
^(T-t-2))/(log((x.*(x-L))^3)))]+(((2.*x-L).*[x.*(x-L)].
^(T-t-1))/(log((x.*(x-L))^2)))-(((2.*x-L).*(T-t).*[x.*(x-L)].
^(T-t-1))/(log(x.*(x-L))))]-x.*k.*(rho-log(x)).*[(x-K).
*(2.*x-L).*[-(((T-t).*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L))))+
(([x.*(x-L)].^(T-t-1))/((log(x.*(x-L)))^2))]-(([x.*(x-L)].
^(T-t))/(log(x.*(x-L))))]+(((rho.*(x-K)).*[x.*(x-L)].^(T-t))/
(log(x.*(x-L))));
plot(t,u,’rd-’,’LineWidth’,2)
hold on
k=1.5;
u=(((-sigma^2)*(x^2))/2).*[[(2.*(x-K))+(2.*x-L)].*[-((T-t).
*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L)))+(([x.*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))]+[(x-K).*(2.*x-L)^2].*[[(-((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))+ (([x*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))].*(T-t-1)-((1)/(log(x.*(x-L)))).
*[(([x.*(x-L)].^(T-t-1))/(log((x.*(x-L))^2)))-(((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))]-(([x.*(x-L)].
^(T-t-2))/(log((x.*(x-L))^3)))]+(((2.*x-L).*[x.*(x-L)].
^(T-t-1))/(log((x.*(x-L))^2)))-(((2.*x-L).*(T-t).*[x.*(x-L)].
^(T-t-1))/(log(x.*(x-L))))]-x.*k.*(rho-log(x)).*[(x-K).
*(2.*x-L).*[-(((T-t).*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L))))+
(([x.*(x-L)].^(T-t-1))/((log(x.*(x-L)))^2))]-(([x.*(x-L)].
^(T-t))/(log(x.*(x-L))))]+(((rho.*(x-K)).*[x.*(x-L)].^(T-t))/
(log(x.*(x-L))));
plot(t,u,’b*--’,’LineWidth’,2)
legend(’k=0.5’,’k=1.0’,’k=1.5’)
hold off


clear
close all
x=15;
K=10;
L=20;
T=1.4;
k=2;
sigma=0.5;
rho=0.1;
t=0:0.2:1.4;
u=(((-sigma^2)*(x^2))/2).*[[(2.*(x-K))+(2.*x-L)].*[-((T-t).
```

```
*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L)))+(([x.*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))]+[(x-K).*(2.*x-L)^2].*[[(-((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))+ (([x*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))].*(T-t-1)-((1)/(log(x.*(x-L)))).
*[(([x.*(x-L)].^(T-t-1))/(log((x.*(x-L))^2)))-(((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))]-(([x.*(x-L)].
^(T-t-2))/(log((x.*(x-L))^3)))]+(((2.*x-L).*[x.*(x-L)].
^(T-t-1))/(log((x.*(x-L))^2)))-(((2.*x-L).*(T-t).*[x.*(x-L)].
^(T-t-1))/(log(x.*(x-L))))]-x.*k.*(rho-log(x)).*[(x-K).
*(2.*x-L).*[-(((T-t).*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L))))+
(([x.*(x-L)].^(T-t-1))/((log(x.*(x-L)))^2))]-(([x.*(x-L)].
^(T-t))/(log(x.*(x-L))))]+(((rho.*(x-K)).*[x.*(x-L)].^(T-t))/
(log(x.*(x-L))));
plot(t,u,’m*-’,’LineWidth’,2)
grid on;
xlabel(’Time(years)’)
ylabel(’Price of the up-and-out European call (u)’)
hold on
k=5;
u=(((-sigma^2)*(x^2))/2).*[[(2.*(x-K))+(2.*x-L)].*[-((T-t).
*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L)))+(([x.*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))]+[(x-K).*(2.*x-L)^2].*[[(-((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))+ (([x*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))].*(T-t-1)-((1)/(log(x.*(x-L)))).
*[(([x.*(x-L)].^(T-t-1))/(log((x.*(x-L))^2)))-(((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))]-(([x.*(x-L)].
^(T-t-2))/(log((x.*(x-L))^3)))]+(((2.*x-L).*[x.*(x-L)].
^(T-t-1))/(log((x.*(x-L))^2)))-(((2.*x-L).*(T-t).*[x.*(x-L)].
^(T-t-1))/(log(x.*(x-L))))]-x.*k.*(rho-log(x)).*[(x-K).
*(2.*x-L).*[-(((T-t).*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L))))+
(([x.*(x-L)].^(T-t-1))/((log(x.*(x-L)))^2))]-(([x.*(x-L)].
^(T-t))/(log(x.*(x-L))))]+(((rho.*(x-K)).*[x.*(x-L)].^(T-t))/
(log(x.*(x-L))));
plot(t,u,’rd-’,’LineWidth’,2)
hold on
k=15;
u=(((-sigma^2)*(x^2))/2).*[[(2.*(x-K))+(2.*x-L)].*[-((T-t).
*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L)))+(([x.*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))]+[(x-K).*(2.*x-L)^2].*[[(-((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))+ (([x*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))].*(T-t-1)-((1)/(log(x.*(x-L)))).
*[(([x.*(x-L)].^(T-t-1))/(log((x.*(x-L))^2)))-(((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))]-(([x.*(x-L)].
^(T-t-2))/(log((x.*(x-L))^3)))]+(((2.*x-L).*[x.*(x-L)].
^(T-t-1))/(log((x.*(x-L))^2)))-(((2.*x-L).*(T-t).*[x.*(x-L)].
```

```
^(T-t-1))/(log(x.*(x-L))))]-x.*k.*(rho-log(x)).*[(x-K).
*(2.*x-L).*[-(((T-t).*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L))))+
(([x.*(x-L)].^(T-t-1))/((log(x.*(x-L)))^2))]-(([x.*(x-L)].
^(T-t))/(log(x.*(x-L))))]+(((rho.*(x-K)).*[x.*(x-L)].^(T-t))/
(log(x.*(x-L))));
plot(t,u,’b*--’,’LineWidth’,2)
legend(’k=2’,’k=5’,’k=15’)
hold off
```

Matlab Codes for plots which shows the price of the up−and−out European call for different values of ρ

```
clear
close all
x=15;
K=10;
L=20;
T=1.4;
k=0.1;
sigma=0.5;
rho=0.01;
t=0:0.2:1.4;
u=(((-sigma^2)*(x^2))/2).*[[(2.*(x-K))+(2.*x-L)].*[-((T-t).
*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L)))+(([x.*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))]+[(x-K).*(2.*x-L)^2].*[[(-((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))+ (([x*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))].*(T-t-1)-((1)/(log(x.*(x-L)))).
*[(([x.*(x-L)].^(T-t-1))/(log((x.*(x-L))^2)))-(((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))]-(([x.*(x-L)].
^(T-t-2))/(log((x.*(x-L))^3)))]+(((2.*x-L).*[x.*(x-L)].
^(T-t-1))/(log((x.*(x-L))^2)))-(((2.*x-L).*(T-t).*[x.*(x-L)].
^(T-t-1))/(log(x.*(x-L))))]-x.*k.*(rho-log(x)).*[(x-K).
*(2.*x-L).*[-(((T-t).*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L))))+
(([x.*(x-L)].^(T-t-1))/((log(x.*(x-L)))^2))]-(([x.*(x-L)].
^(T-t))/(log(x.*(x-L))))]+(((rho.*(x-K)).*[x.*(x-L)].^(T-t))/
(log(x.*(x-L))));
plot(t,u,’m*-’,’LineWidth’,2)
grid on;
xlabel(’Time(years)’)
ylabel(’Price of the up-and-out European call (u)’)
hold on
rho=0.05;
u=(((-sigma^2)*(x^2))/2).*[[(2.*(x-K))+(2.*x-L)].*[-((T-t).
```

```
*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L)))+(([x.*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))]+[(x-K).*(2.*x-L)^2].*[[(-((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))+ (([x*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))].*(T-t-1)-((1)/(log(x.*(x-L)))).
*[(([x.*(x-L)].^(T-t-1))/(log((x.*(x-L))^2)))-(((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))]-(([x.*(x-L)].
^(T-t-2))/(log((x.*(x-L))^3)))]+(((2.*x-L).*[x.*(x-L)].
^(T-t-1))/(log((x.*(x-L))^2)))-(((2.*x-L).*(T-t).*[x.*(x-L)].
^(T-t-1))/(log(x.*(x-L))))]-x.*k.*(rho-log(x)).*[(x-K).
*(2.*x-L).*[-(((T-t).*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L))))+
(([x.*(x-L)].^(T-t-1))/((log(x.*(x-L)))^2))]-(([x.*(x-L)].
^(T-t))/(log(x.*(x-L))))]+(((rho.*(x-K)).*[x.*(x-L)].^(T-t))/
(log(x.*(x-L))));
plot(t,u,’rd-’,’LineWidth’,2)
hold on
rho=0.2;
u=(((-sigma^2)*(x^2))/2).*[[(2.*(x-K))+(2.*x-L)].*[-((T-t).
*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L)))+(([x.*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))]+[(x-K).*(2.*x-L)^2].*[[(-((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))+ (([x*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))].*(T-t-1)-((1)/(log(x.*(x-L)))).
*[(([x.*(x-L)].^(T-t-1))/(log((x.*(x-L))^2)))-(((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))]-(([x.*(x-L)].
^(T-t-2))/(log((x.*(x-L))^3)))]+(((2.*x-L).*[x.*(x-L)].
^(T-t-1))/(log((x.*(x-L))^2)))-(((2.*x-L).*(T-t).*[x.*(x-L)].
^(T-t-1))/(log(x.*(x-L))))]-x.*k.*(rho-log(x)).*[(x-K).
*(2.*x-L).*[-(((T-t).*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L))))+
(([x.*(x-L)].^(T-t-1))/((log(x.*(x-L)))^2))]-(([x.*(x-L)].
^(T-t))/(log(x.*(x-L))))]+(((rho.*(x-K)).*[x.*(x-L)].^(T-t))/
(log(x.*(x-L))));
plot(t,u,’b*--’,’LineWidth’,2)
legend(’rho=0.01’,’rho=0.05’,’rho=0.20’)
hold off


clear
close all
x=15;
K=10;
L=20;
T=1.4;
k=0.1;
sigma=0.5;
rho=2;
t=0:0.2:1.4;
u=(((-sigma^2)*(x^2))/2).*[[(2.*(x-K))+(2.*x-L)].*[-((T-t).
```

```
*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L)))+(([x.*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))]+[(x-K).*(2.*x-L)^2].*[[(-((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))+ (([x*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))].*(T-t-1)-((1)/(log(x.*(x-L)))).
*[(([x.*(x-L)].^(T-t-1))/(log((x.*(x-L))^2)))-(((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))]-(([x.*(x-L)].
^(T-t-2))/(log((x.*(x-L))^3)))]+(((2.*x-L).*[x.*(x-L)].
^(T-t-1))/(log((x.*(x-L))^2)))-(((2.*x-L).*(T-t).*[x.*(x-L)].
^(T-t-1))/(log(x.*(x-L))))]-x.*k.*(rho-log(x)).*[(x-K).
*(2.*x-L).*[-(((T-t).*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L))))+
(([x.*(x-L)].^(T-t-1))/((log(x.*(x-L)))^2))]-(([x.*(x-L)].
^(T-t))/(log(x.*(x-L))))]+(((rho.*(x-K)).*[x.*(x-L)].^(T-t))/
(log(x.*(x-L))));
plot(t,u,’m*-’,’LineWidth’,2)
grid on;
xlabel(’Time(years)’)
ylabel(’Price of the up-and-out European call (u)’)
hold on
rho=3;
u=(((-sigma^2)*(x^2))/2).*[[(2.*(x-K))+(2.*x-L)].*[-((T-t).
*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L)))+(([x.*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))]+[(x-K).*(2.*x-L)^2].*[[(-((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))+ (([x*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))].*(T-t-1)-((1)/(log(x.*(x-L)))).
*[(([x.*(x-L)].^(T-t-1))/(log((x.*(x-L))^2)))-(((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))]-(([x.*(x-L)].
^(T-t-2))/(log((x.*(x-L))^3)))]+(((2.*x-L).*[x.*(x-L)].
^(T-t-1))/(log((x.*(x-L))^2)))-(((2.*x-L).*(T-t).*[x.*(x-L)].
^(T-t-1))/(log(x.*(x-L))))]-x.*k.*(rho-log(x)).*[(x-K).
*(2.*x-L).*[-(((T-t).*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L))))+
(([x.*(x-L)].^(T-t-1))/((log(x.*(x-L)))^2))]-(([x.*(x-L)].
^(T-t))/(log(x.*(x-L))))]+(((rho.*(x-K)).*[x.*(x-L)].^(T-t))/
(log(x.*(x-L))));
plot(t,u,’rd-’,’LineWidth’,2)
hold on
rho=4;
u=(((-sigma^2)*(x^2))/2).*[[(2.*(x-K))+(2.*x-L)].*[-((T-t).
*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L)))+(([x.*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))]+[(x-K).*(2.*x-L)^2].*[[(-((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))+ (([x*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))].*(T-t-1)-((1)/(log(x.*(x-L)))).
*[(([x.*(x-L)].^(T-t-1))/(log((x.*(x-L))^2)))-(((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))]-(([x.*(x-L)].
^(T-t-2))/(log((x.*(x-L))^3)))]+(((2.*x-L).*[x.*(x-L)].
^(T-t-1))/(log((x.*(x-L))^2)))-(((2.*x-L).*(T-t).*[x.*(x-L)].
```

```
^(T-t-1))/(log(x.*(x-L))))]-x.*k.*(rho-log(x)).*[(x-K).
*(2.*x-L).*[-(((T-t).*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L))))+
(([x.*(x-L)].^(T-t-1))/((log(x.*(x-L)))^2))]-(([x.*(x-L)].
^(T-t))/(log(x.*(x-L))))]+(((rho.*(x-K)).*[x.*(x-L)].^(T-t))/
(log(x.*(x-L))));
plot(t,u,’b*--’,’LineWidth’,2)
legend(’rho=2’,’rho=3’,’rho=4’)
hold off
```

Matlab Codes for plots which shows the price of the up−and−out European call for different values of σ

```
clear
close all
x=15;
K=10;
L=20;
T=1.4;
k=0.5;
sigma=0.02;
rho=0.1;
t=0:0.2:1.4;
u=(((-sigma^2)*(x^2))/2).*[[(2.*(x-K))+(2.*x-L)].*[-((T-t).
*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L)))+(([x.*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))]+[(x-K).*(2.*x-L)^2].*[[(-((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))+ (([x*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))].*(T-t-1)-((1)/(log(x.*(x-L)))).
*[(([x.*(x-L)].^(T-t-1))/(log((x.*(x-L))^2)))-(((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))]-(([x.*(x-L)].
^(T-t-2))/(log((x.*(x-L))^3)))]+(((2.*x-L).*[x.*(x-L)].
^(T-t-1))/(log((x.*(x-L))^2)))-(((2.*x-L).*(T-t).*[x.*(x-L)].
^(T-t-1))/(log(x.*(x-L))))]-x.*k.*(rho-log(x)).*[(x-K).
*(2.*x-L).*[-(((T-t).*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L))))+
(([x.*(x-L)].^(T-t-1))/((log(x.*(x-L)))^2))]-(([x.*(x-L)].
^(T-t))/(log(x.*(x-L))))]+(((rho.*(x-K)).*[x.*(x-L)].^(T-t))/
(log(x.*(x-L))));
plot(t,u,’m*-’,’LineWidth’,2)
grid on;
xlabel(’Time(years)’)
ylabel(’Price of the up-and-out European call (u)’)
hold on
sigma=0.25;
u=(((-sigma^2)*(x^2))/2).*[[(2.*(x-K))+(2.*x-L)].*[-((T-t).
```

```
*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L)))+(([x.*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))]+[(x-K).*(2.*x-L)^2].*[[(-((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))+ (([x*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))].*(T-t-1)-((1)/(log(x.*(x-L)))).
*[((([x.*(x-L)].^(T-t-1))/(log((x.*(x-L))^2)))-(((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))]-(([x.*(x-L)].
^(T-t-2))/(log((x.*(x-L))^3)))]+(((2.*x-L).*[x.*(x-L)].
^(T-t-1))/(log((x.*(x-L))^2)))-(((2.*x-L).*(T-t).*[x.*(x-L)].
^(T-t-1))/(log(x.*(x-L))))]-x.*k.*(rho-log(x)).*[(x-K).
*(2.*x-L).*[-(((T-t).*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L))))+
(([x.*(x-L)].^(T-t-1))/((log(x.*(x-L)))^2))]-(([x.*(x-L)].
^(T-t))/(log(x.*(x-L))))]+(((rho.*(x-K)).*[x.*(x-L)].^(T-t))/
(log(x.*(x-L))));
plot(t,u,’rd-’,’LineWidth’,2)
hold on
sigma=0.5;
u=(((-sigma^2)*(x^2))/2).*[[(2.*(x-K))+(2.*x-L)].*[-((T-t).
*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L)))+(([x.*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))]+[(x-K).*(2.*x-L)^2].*[[(-((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))+ (([x*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))].*(T-t-1)-((1)/(log(x.*(x-L)))).
*[((([x.*(x-L)].^(T-t-1))/(log((x.*(x-L))^2)))-(((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))]-(([x.*(x-L)].
^(T-t-2))/(log((x.*(x-L))^3)))]+(((2.*x-L).*[x.*(x-L)].
^(T-t-1))/(log((x.*(x-L))^2)))-(((2.*x-L).*(T-t).*[x.*(x-L)].
^(T-t-1))/(log(x.*(x-L))))]-x.*k.*(rho-log(x)).*[(x-K).
*(2.*x-L).*[-(((T-t).*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L))))+
(([x.*(x-L)].^(T-t-1))/((log(x.*(x-L)))^2))]-(([x.*(x-L)].
^(T-t))/(log(x.*(x-L))))]+(((rho.*(x-K)).*[x.*(x-L)].^(T-t))/
(log(x.*(x-L))));
plot(t,u,’b*--’,’LineWidth’,2)
legend(’sigma=0.20’,’sigma=0.25’,’sigma=0.50’)
hold off


clear
close all
x=15;
K=10;
L=20;
T=1.4;
k=0.5;
sigma=2;
rho=0.1;
t=0:0.2:1.4;
u=(((-sigma^2)*(x^2))/2).*[[(2.*(x-K))+(2.*x-L)].*[-((T-t).
```

```
*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L)))+(([x.*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))]+[(x-K).*(2.*x-L)^2].*[[(-((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))+ (([x*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))].*(T-t-1)-((1)/(log(x.*(x-L)))).
*[((([x.*(x-L)].^(T-t-1))/(log((x.*(x-L))^2)))-(((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))]-(([x.*(x-L)].
^(T-t-2))/(log((x.*(x-L))^3)))]+(((2.*x-L).*[x.*(x-L)].
^(T-t-1))/(log((x.*(x-L))^2)))-(((2.*x-L).*(T-t).*[x.*(x-L)].
^(T-t-1))/(log(x.*(x-L))))]-x.*k.*(rho-log(x)).*[(x-K).
*(2.*x-L).*[-(((T-t).*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L))))+
(([x.*(x-L)].^(T-t-1))/((log(x.*(x-L)))^2))]-(([x.*(x-L)].
^(T-t))/(log(x.*(x-L))))]+(((rho.*(x-K)).*[x.*(x-L)].^(T-t))/
(log(x.*(x-L))));
plot(t,u,’m*-’,’LineWidth’,2)
grid on;
xlabel(’Time(years)’)
ylabel(’Price of the up-and-out European call (u)’)
hold on
sigma=5;
u=(((-sigma^2)*(x^2))/2).*[[(2.*(x-K))+(2.*x-L)].*[-((T-t).
*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L)))+(([x.*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))]+[(x-K).*(2.*x-L)^2].*[[(-((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))+ (([x*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))].*(T-t-1)-((1)/(log(x.*(x-L)))).
*[((([x.*(x-L)].^(T-t-1))/(log((x.*(x-L))^2)))-(((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))]-(([x.*(x-L)].
^(T-t-2))/(log((x.*(x-L))^3)))]+(((2.*x-L).*[x.*(x-L)].
^(T-t-1))/(log((x.*(x-L))^2)))-(((2.*x-L).*(T-t).*[x.*(x-L)].
^(T-t-1))/(log(x.*(x-L))))]-x.*k.*(rho-log(x)).*[(x-K).
*(2.*x-L).*[-(((T-t).*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L))))+
(([x.*(x-L)].^(T-t-1))/((log(x.*(x-L)))^2))]-(([x.*(x-L)].
^(T-t))/(log(x.*(x-L))))]+(((rho.*(x-K)).*[x.*(x-L)].^(T-t))/
(log(x.*(x-L))));
plot(t,u,’rd-’,’LineWidth’,2)
hold on
sigma=15;
u=(((-sigma^2)*(x^2))/2).*[[(2.*(x-K))+(2.*x-L)].*[-((T-t).
*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L)))+(([x.*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))]+[(x-K).*(2.*x-L)^2].*[[(-((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))+ (([x*(x-L)].^(T-t-1))/
(log((x.*(x-L))^2)))].*(T-t-1)-((1)/(log(x.*(x-L)))).
*[((([x.*(x-L)].^(T-t-1))/(log((x.*(x-L))^2)))-(((T-t).
*[x.*(x-L)].^(T-t-2))/(log(x.*(x-L))))]-(([x.*(x-L)].
^(T-t-2))/(log((x.*(x-L))^3)))]+(((2.*x-L).*[x.*(x-L)].
^(T-t-1))/(log((x.*(x-L))^2)))-(((2.*x-L).*(T-t).*[x.*(x-L)].
```

```
^(T-t-1))/(log(x.*(x-L))))]-x.*k.*(rho-log(x)).*[(x-K).
*(2.*x-L).*[-(((T-t).*[x.*(x-L)].^(T-t-1))/(log(x.*(x-L))))+
(([x.*(x-L)].^(T-t-1))/((log(x.*(x-L)))^2))]-(([x.*(x-L)].
^(T-t))/(log(x.*(x-L))))]+(((rho.*(x-K)).*[x.*(x-L)].^(T-t))/
(log(x.*(x-L))));
plot(t,u,’b*--’,’LineWidth’,2)
legend(’sigma=2’,’sigma=5’,’sigma=15’)
hold off
```