OPTIMAL PORTFOLIO MANAGEMENT WHEN STOCKS ARE DRIVEN BY MEAN-REVERTING PROCESSES

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M.Sc. (Mathematical Modelling) Dissertation
University of Dar es Salaam University of Dar es Salaam

September, 2012

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By

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A Dissertation Submitted in Partial Fulfilment of the Requirements for the Degree of Master of Science(Mathematical Modelling) of the University of Dar es Salaam

University of Dar es Salaam

September, 2012

CERTIFICATION

The undersigned certify that they have read and hereby recommend for acceptance by the University of Dare es Salaam the dissertation entitled: ***Optimal Portfolio Management when Stocks are driven by Mean-Reverting Processes***, in partial fulfillment of the requirements for the degree of Master of Science (Mathematical modelling) of the University of Dar es Salaam.

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Date: 02/08/2012

DECLARATION AND COPYRIGHT

I, **MBIGILI, Lusungu Julius**, declare that this dissertation is my own original work and that it has never been presented and will never be presented to any other University for a similar or any other degree award.

Signature: ________________

ACKNOWLEDGEMENTS

First of all, I stand humbly with all my heartfelt thanks to you GOD for making this possible. It's by Your grace that I have gone this far.

My special thanks to Belgian government for the scholarship through Belgian Technical Cooperation(BTC) which enabled me to undertake this study. For sure without their sponsorship I could have not so far achieved this level. My sincere appreciation goes to Norwegian people for granting this NOMA programme and the University of Dar es Salaam for hosting it.

I wish to extend my heartfelt gratitude to my supervisors Dr.Sure Mataramvura from the university of Capetown(South Africa) and Dr.Charles W. Mahera from the university of Dar es Salaam(Tanzania) for their valuable support, constructive ideas, tireless effort, continuous encouragement and the time they spent reading my work, giving constructive criticism and guidance on how to move in the right direction. From them, indeed I have learned a lot and also how to work independently.

Special appreciation goes to Professor E.Massawe, the head of mathematics department(University of Dar es Salaam) for his constant support, continuous encouragement and the mentor role he played throughout my graduate studies. This would be a lack of respect if I don't recognize the other staff members of this department for their valuable support and encouragement. It's them who made this place favorable, warmest and excellent with welcoming ambiance for knowledge acquisition.

I would like to express my sincere gratitude to Ardhi university administration(my employer) for providing me all the support and the room to pursue my masters degree studies. Special thanks also to dean of the School of geospatial Sciences and technology(SGST)of Ardhi university Dr.Liwa and heads of departments in the school Dr.Chaula J. and Dr.Haghai M. for their valuable support and continuous encouragements throughout this period of my graduate studies. Special appreciation to my workmates Ali Ahmada and Edwin Mvanda for their constructive ideas, constant support and continuous encouragement toward completion of my dissertation. I would also like to recognize the moral support and encouragement from other fellow staff members in SGST more particularly Julieth Joseph, she was always there whenever I needed her assistance, and with all my heart, I truly appreciate that.

Warmest thanks to all my fellow classmates for their presence whenever I needed help and the mutual encouragement and love we lived and shared. We were firmly together in good and tough moments. All credits to my beloved friend and classmate the late Josephine Kaleso for her lovely words of encouragement, support and wishes. She always wanted to see me excelling academically and further going beyond this level. Her words that I kept in my heart truly keep me going and progressing on and indeed today I have reached this far. I wish she could be around to celebrate this success, but God loved her most. Special tributes also to our comrades who lost their lives in the road accident. I personally recognize their valuable contributions in this success that I have attained. They truly did not leave this planet vainly, yet their words of wisdom still in our hearts and make us move on. May the Almighty God grant their souls the eternal life.

The last but not least; I would like to express my utmost gratitude to my mother, my children Priscilla and Prosper, my brothers Baraka, Fredy and Alpha and my sisters Deborah, Vaireth together with their families as well as my little sister Emmy for their colorful love, constant support and continuous encouragement throughout the period of my studies. I have truly experienced your virtual presence all the time I needed you and really I appreciate that from the bottom of my heart.

Finally, I agree to take the responsibility for any shortcomings that might be discovered in this work and therefore it should not be attributed to any one of the acknowledged individuals.

DEDICATION

To you my beloved mother Merina Timoth Kisoma, my beloved children Priscilla and Prosper and my beloved friend, the late Josephine Kaleso, I dedicate this dissertation.

ABSTRACT

In this work, we present and solve the problem of portfolio optimization within the context of continuous-time stochastic model of financial variables. We consider an investment problem where an investor has two assets, namely, risk-free assets(eg bonds) and risky assets(eg stocks) to invest on and tries to maximize the expected utility of the wealth at some future time τ. The evolution of the risk-free asset is described deterministically while the dynamics of the risky asset is described by the geometric mean reversion(GMR) model.

The controlled wealth stochastic differential equation(SDE) and the portfolio problem are formulated. The portfolio optimization problem is then successfully formulated and solved with the help of the theory of stochastic control technique where the dynamic programming principle(DPP) and the HJB theory were used. We obtained very interesting results which are the solution of the non-linear second order partial differential equation and the optimal policy which is the optimal control strategy for the investment process.

So far we have considered utility functions which are members of hyperbolic absolute risk aversion(HARA) family, called power and exponential utility. In both cases, the optimal control(investment strategy) has explicit form and is wealth dependant, in the sense that, as the investor becomes more rich, the less he invests on the risky assets. Linearization of the logarithmic term in the portfolio problem was necessary to be undertaken for making the work of obtaining the explicit form of the optimal control much simple than it was expected.

LIST OF FIGURES

LIST OF ABBREVIATIONS

ADP: Approximate dynamic programming BSDE: Backward stochastic differential equation

DPE: Dynamic programming equation

DPP: Dynamic programming principle

GMR: Geometric mean reversion

HARA: Hyperbolic absolute risk aversion

HJB: Hamilton-Jacobi-Bellman

HJBI: Hamilton-Jacobi-Bellman-Isaacs

MPT: Modern portfolio theory

PDE: Partial differential equation

SDE: Stochastic differential equation

SGST: School of geospatial science and technology

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CHAPTER ONE
INTRODUCTION

1.1 General background

The application of control engineering methods and techniques to financial problems is called financial engineering. Financial engineering is therefore regarded as the use of mathematical finance and modelling to make pricing, hedging, trading and portfolio management decisions. Our main concern is all about portfolio management decisions. By definition, portfolio is the collection of investments held by an institution or an individual. Holding a portfolio with different investments instead of a single one is reducing the investor's risk. Conventional wisdom has always dictated "*not putting all your eggs into one basket*". In more technical term, this old adage is addressing the benefits of the so-called diversification principle(Fabozzi *et al.*, 2006, and Keel, 2006).

The practice of portfolio management should be thought of as how the investor should go about actually managing their wealth or having it managed for them. Having a framework within which to make portfolio decisions allows investors to be more consistent in managing their portfolios. Portfolio management therefore involves a series of decisions and actions that are made by every investor, whether individual or institution. This provides investors with an organized, systematic framework for managing a portfolio. Portfolio management process has therefore been described as "*the second step in the investment decision process, involving the management of a group of assets, i.e. a portfolio as a unit*". Consider portfolio management as an ongoing process by which:

(i) each investor identifies objectives, constraints and preferences as part of an orderly framework to guide them in managing their portfolios

(ii) capital market expectations for the economy, industries and sectors and individual securities are considered and quantified

(iii) strategies are developed and implemented. This involves asset allocation, portfolio optimization and selection of securities

(iv) portfolio factors are monitored and responses are made as investor objectives and constraints and/or market expectations change

(v) the portfolio is rebalanced as necessary by repeating the asset allocation, portfolio strategy and security selection step and

(vi) portfolio performance is measured and evaluated to ensure attainment of the investor objectives(Jones, 2007).

Our main concern is about portfolio optimization which is part of development and implementation of the portfolio strategies in the portfolio management process. Portfolio optimization problems within the context of continuous-time stochastic models of financial variables have been widely studied in the field of mathematical finance and engineering(Luo *et al.*, 2011). In general, the problem of portfolio construction is about investing our own money in financial assets such that certain requirements regarding the expected profits(expected returns) and possible losses(possible risks) are matched. And therefore, *optimization problem* involves solving a problem by finding the best solution of objective function from a set of all possible solutions.

From the set of available assets to invest on, with prices and distribution of their returns, the question is, what is the optimal portfolio? This is the common problem always triggering the investor's mind as far as the investment is concerned. In mathematical finance, the idea behind portfolio optimization is choosing the best strategy when an investor is faced with different investment decisions in regard to their wealth. An investor, dynamically allocates wealth between the risky and the risk-free assets with the goal of maximizing total expected returns while minimizing the variance, i.e. the possible risk. In order to reduce the risk and increase the profit, the investor may invest his assets in appropriate percentage in different markets. The investor indeed wants to attain maximum profit by choosing appropriate investment strategy, i.e. an optimal strategy and if it exists, will depend on some factors such as: (i)market information (ii)investor's initial wealth (iii)investors belief and behaviour in-front of the market risks and (iv)the criterion by which the investor decides whether a strategy is good or not(Khati, 2011).

The concept of optimization is fundamental to financial theory. Portfolio optimization was first introduced by Harry Markowitz in his seminal work "*Portfolio selection*"in the journal of finance in which he used mean variance analysis to show that a portfolio is fully characterized by its return and variance. He introduced what is known as the Markowitz frontier(efficient frontier), the set of all efficient portfolios in which the investor is restricted to choose his portfolio according to initial wealth(Markowitz, 1952). The portfolio that belongs to the efficient frontier is the one offering the minimum level of risk and as such, is the best portfolio in the Markowitz's model. However, it has been suggested by many researchers that, multi-period approach may provide superior performance over the single-period approach(Herzog, 2005 and Fabozzi *et al.*, 2006).

Merton(1969, 1971), the nobel prize of economic, in the case of continuous-time, by his method of optimal stochastic control, explicitly solved the problem of optimal portfolio allocation in a market with a risk-free asset and a stock as investment alternatives(Benth *et al.*, 2003). He used also HJB equation to derive a

PDE and produce solutions for both finite and infinite horizon using exponential and logarithmic utility functions.

While paying attention to continuous time portfolio optimization problem, researchers have as well noted the impact of mean-reversion on optimal portfolio choice and also is of central importance in the asset allocation problem. The random walk model was the first to be in place as the basic model of stock prices based on the assumption of market efficiency. The basic idea is that returns can be represented as unforecastable fluctuations around some mean return. This assumption implies that the distribution of the returns at time t is independent from, or at least uncorrelated with, the distribution of returns in previous moments. Therefore, mean-reversion is thought of as a modification of the random walk, where returns change are not completely independent of one another but rather are related. Generally, mean-reversion is considered as the change of market return in the direction of reversion level as the reaction to prior change in the market return. After a positive change in the actual price, mean-reversion causes a negative subsequent change and vice versa. Mean-reversion has actually received a considerable attention in the financial world as a classic indicator of predictability in financial markets and has more economic logic than geometric Brownian model(Dmitrasinovic-Vidovic *et al.*, 2011, Dung, 2011, Emmer *et al.*, 2004, Koijen *et al.*, 2009, Munk *et al.*, 2004, and Poterba *et al.*, 1988).

In general, the problem of portfolio optimization can be successfully solved by the theory of stochastic optimal control, where Dynamic programming principle(DPP) and HJB theory are instrumental for finding a solution. The optimal control theory states the mathematical conditions of optimality given the dynamic model that describes the system and the objective function we want to optimize. Thus, by considering an optimal control of Ito-type processes which satisfy the stochastic differential equation(SDE) w.r.t some Wiener process, our goal is to choose the investment control strategy(i.e dynamic portfolio strategy) to maximize the expected utility of wealth at some future time τ (Fleming *et al.*, 2004, Øksendal, 2003, Ross, 2008).

1.2 Statement of the problem

We mainly focus on portfolio problem of an investor who trades continuously from say time t and maximizes expected utility of wealth at some future time $t_1 > t$. The problem of finding the optimal strategy is classical and has been extensively studied. Most of these studies considered stock price as Markov process. Benth *et al.*(2003) through the study of optimal portfolio optimization for an investor who can trade in a risk-free bond and stock, included the stochastic volatility in the dynamics of the risky asset. Its drawback is that, volatility is not directly observable in the market unlike the stock price, and it is therefore in practice impossible to follow portfolio rules where one must take the level of volatility explicitly into account.

The recent study of Dmitrasinovic-Vidovic *et al.*(2011) investigated the portfolio selection consisting of instruments whose logarithms are mean-reverting. They assumed that portfolios are constant and also short-selling and borrowing are allowed, and the optimal strategies were found in the sense of time-independent portfolios, i.e. portfolios which do not depend on asset prices, which is not the case in real life situation.

In our study, we shall focus on optimal strategies in the sense that portfolio depends on asset prices and no borrowing and short-selling(thats no inflow and external flow of cash). As previous observations might be useful in predicting the future prices of the risky asset, then stock-price indexes can be characterized as mean-reversion processes(Chaudhuri *et al.*, 2004). Therefore,we shall consider the dynamics of the risky asset described by the geometric mean reversion(GMR) model.

1.3 Research objectives

The main objective is to optimize the portfolio value by maximizing the expected utility of wealth at some future time τ when stocks are driven by mean-reverting processes.

The specific objectives of this study are:

(i) To formulate the dynamic portfolio optimization model using suitable utility function of the total wealth and the Markov control policy.

(ii) To determine the optimal control strategy which maximizes the investors expected utility of wealth at some future time τ.

1.4 Significance of the study

The concept of optimization is fundamental to finance theory. The seminal-work of Harry Markowitz demonstrated that financial decision-making for a rational agent is essentially a question of achieving an optimal trade-off between risk and returns(Fabozzi *et al.*, 2006). Economics and finance are the most important fields of application of stochastic control theory which deals with the optimal portfolio problems. In everyday life, investors are looking for optimal strategic investments, where they can diversify their wealth in risk-free and risky assets with the hope of attaining maximum profit with minimum risk to take on.

Therefore, in this study of optimal portfolio management, we wish to achieve the results which will be useful and applicable in optimal portfolio diversification, where investors are interested in looking for portfolios with maximum returns for a given level of risk. Today in many firms, portfolio management remains a purely judgemental process based on qualitative and not quantitative assessment as it was suggested by Harry Markowitz(1952) in his famous paper. The study will also help investors to manage their portfolios based on qualitative assessment more particulary in portfolio rebalancing.

1.5 Research hypothesis

(i) There is a Markov control process and a suitable utility function of wealth from which the optimal portfolio model is formulated.

(ii) There is an optimal control policy for which the value function(optimal performance) is attained for some time τ.

1.6 Research methodology

Optimal portfolio problems have been widely studied and Stochastic optimal control technique has been and continue to be the most applicable tool in handling such problems. In the derivation of the optimal strategies(or rules) we shall use the technique of stochastic dynamic programming. The dynamic programming principle (DPP) and Hamilton-Jacobi-Bellman (HJB) theory are the tools that we shall employ in the formulation of optimal portfolio problem. Maximizing the expected utility of wealth in the future time, we shall apply the DPP to derive the HJB equation which is often known as non-linear PDE.

Solving that kind of PDE with the general utility function would be a tedious work and almost impossible and for that reason we shall use some specific utility functions, i.e. *power utility and exponential utility* to obtain the value function which is the solution for the PDE. The value function needs to be smooth in order to apply the classical form of Ito's rule. Also, because of the nature of the risky asset model which is the GMR model, the linearization of the logarithmic term in the HJB equation shall be done by Taylor expansion technique. This would help in obtaining the optimal rules explicitly.

Finally, we shall employ the software called MATLAB to analyze graphically the behaviour of the optimal policy with respect to the wealth and that of the value function with respect to time and the wealth with the same market parameters.

CHAPTER TWO
GENERAL PRESENTATION AND LITERATURE REVIEW

2.1 General presentation

We initially start this chapter by introducing some preliminary definitions and results that will be useful throughout this work.

2.1.1 Preliminaries

Most of the following preliminaries and the rest of the general presentation, mainly come from Karatzas *et al*(1988), Pham(2009), Øksendal(2003) and Yong *et al*(1999), and since they are just briefly discussed, then one can consult the suggested literature for more details. In this work we consider continuous-time stochastic processes, and the time interval $\mathcal{T} = [0, T]$, where $0 < T < \infty$. Actually stochastic process is a family $X = (X_t)_{t \in \mathcal{T}}$ of random variables defined on the probability space $(\Omega, \mathcal{F}, P)$ and valued in a measurable space $\mathbb{R}$ and indexed by time t. For each $\omega \in \Omega$, the mapping $X(\omega) : t \in \mathcal{T} \to X(t, \omega)$ is called the path of the process for the event ω.

Definition 1 (Filtration) *A filtration on a probability space* $(\Omega, \mathcal{F}, P)$ *is an increasing family* $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,\infty]}$ *of* σ*-fields of* $\mathcal{F} : \mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$, $\forall\ 0 \leq s \leq t$ *in* $\mathcal{T}$

$\mathcal{F}_t$ is interpreted as the information known at time t, and increases as time elapses. The quadruple $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is called filtered probability space.

We say that a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \in \mathcal{T}}$ satisfies the usual conditions if it is right continuous, i.e.

$\mathcal{F}_{t^+} = \cap_{s \geq t} \mathcal{F}_s = \mathcal{F}_t, \forall\ t \in \mathcal{T}$

and if it is complete, i.e. $\mathcal{F}_0$ contains the negligible set of $\mathcal{F}_{\bar{\mathcal{T}}} = \sigma(\cup_{t\in\mathcal{T}}\mathcal{F}_t)$(The smallest σ-field containing all $\mathcal{F}_t, t \in \mathcal{T}$).

Definition 2 (Adapted process) *A process $(X_t)_{t\in\mathcal{T}}$ is adapted (with respect to $\mathbb{F}$) if $\forall\ t \in \mathcal{T}$, X_t is $\mathcal{F}_t$-measurable.*

Thus an adapted process is a process whose value at any instant t is revealed by the information $\mathcal{F}_t$. Sometimes the process is known as non-anticipative.

Definition 3 (Progressively measurable) *A process $(X_t)_{t\in\mathcal{T}}$ is progressively measurable(with respect to $\mathbb{F}$) if for any time $t \in \mathcal{T}$, the mapping $(s,\omega) \mapsto X(s,\omega)$ is measurable on $[0,t] \times \Omega$ equipped with product σ-field $\mathscr{B}([0,T]) \otimes \mathcal{F}_t$ ($\mathbb{F}$-progressively measurable).*

Thus from the basic interpretation of $\mathcal{F}_t$ as the available information up to time t, one would be interested to know if an event characterized by its first arrival time $\tau(\omega)$, occurred or not before time t given the observation in $\mathcal{F}_t$. This leads to the notion of stopping time.

Actually, a random time T is an $\mathcal{F}$-measurable random variable with values in $[0,\infty]$(Karatzas *et al*(1988)).

Definition 4 *If X is a stochastic process and T is a random time, then the function X_T on the event $\{T < \infty\}$ is defined by*

$$X_T(\omega) \triangleq X_{T(\omega)}(\omega).$$

If $X_\infty(\omega)$ is defined for all $\omega \in \Omega$, then X_T can also be defined on Ω by setting

$$X_T(\omega) \triangleq X_\infty(\omega) \quad \text{on} \quad \{T = \infty\}.$$

Definition 5 (Stopping time) *A random variable $\tau : \Omega \to [0,\infty]$, i.e a random time is a stopping time(with respect to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathcal{T}}$) if $\forall\ t \in \mathcal{T}$ $\{\tau \le t\} = \{\omega \in \Omega : \tau(\omega) \le t\} \in \mathcal{F}_t$.*

Given a process $\{X_t\}_{t\in\mathcal{T}}$ and a stopping time τ, we define the random variable X_τ on $\{\tau \in \mathcal{T}\}$ by

$$X_\tau(\omega) \triangleq X_{\tau(\omega)}(\omega).$$

We see that if X is measurable then X_τ is a random variable on $\{\tau \in \mathcal{T}\}$. We then introduce the stopped process(at τ) X^τ defined by

$$X_t^\tau \triangleq X_{\tau\wedge t} \qquad t \in \mathcal{T}.$$

Theorem 1 *Let $X = \{X_t)_{t\in\mathcal{T}}, \mathcal{F}_t\}$ be a progressively measurable process, and let τ be a stopping time of the filtration $\{\mathcal{F}_t\}$. Then the random variable $X_\tau \mathbb{I}_{\{\tau<\infty\}}$ is $\mathcal{F}_\tau$-measurable and the stopped process $\{X_{\tau\wedge t}, \mathcal{F}_t; t \in \mathcal{T}\}$ is progressively measurable.*

The proof of this theorem is found in Karatzas *et al*(1988). The stochastic process X is said to be **Cad-lag**(resp.continuous) if for each $\omega \in \Omega$, the path $X(\omega)$ is right continuous and admits a left limit(resp. is continuous).

Theorem 2 *Let X be a cad-lag, adapted process and $\mathcal{G}$ be an open subset of $\mathbb{R}$*

1. *If the filtration $\mathbb{F}$ satisfies the usual conditions, then the hitting time of $\mathcal{G}$ defined by $\tau_{\mathcal{G}} = \inf\{t \geq 0 : X_t \in \mathcal{G}\}$ (with the convection $\inf \emptyset = \infty$) is a stopping time.*
2. *If X is continuous, then the exit time of $\mathcal{G}$ defined by $\tau_{\mathcal{G}} = \inf\{t \geq 0 : X_t \notin \mathcal{G}\}$ is a predictable stopping time.*

A standard Brownian motion is a stochastic process $(W_t)_t$ with values in $\mathbb{R}$ defined for $t \in [0, \infty)$ such that

1. $W_0 = 0$ almost surely.
2. The sample paths $t \mapsto W_t$ are a.s continuous.
3. The increments are normally distributed i.e $W_t - W_s \sim \mathcal{N}(0, t-s)$ for any $0 < s < t$.

4. The increments are independent, i.e. $W_u - W_v$ and $W_t - W_s$ are independent for any $0 \leq v \leq u \leq s \leq t$.

Definition 6 (Brownian motion with respect to a filtration. Pham, 2009) *A standard Brownian motion* $W = (W_t)_{t\in\mathcal{T}}$ *on* $\mathcal{T}$ *with respect to a filtration* $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathcal{T}}$ *is a continuous* $\mathbb{F}$*-adapted process and valued in* $\mathbb{R}$ *such that*

1. $W_0 = 0$
2. $\forall\ 0 \leq s \leq t$ *in* $\mathcal{T}$*, the increment* $W_t - W_s$ *is independent of* $\mathcal{F}_s$ *and follows a centered Gaussian distribution, i.e.* $W_t - W_s \sim \mathcal{N}(0, t-s)$.

However, we would rather like to impose some conditions on $b(t, X(t))$ and $\sigma(t, X(t))$ to guarantee the existence and uniqueness of the strong solution of any stochastic differential equation.

Definition 7 (Lin, 2006) *A Borel measurable function* $f(t,x)$ *from* $\mathcal{T} \times \mathbb{R}$ *into* $\mathbb{R}$ *is said to satisfy global Lipschitz condition in* x *if there exists a constant* $k > 0$ *such that*

$$|f(t,x) - f(t,y)| \leq k\,|x-y| \qquad \forall\ t \in \mathcal{T}; \qquad x, y \in \mathbb{R}. \tag{2.1}$$

Definition 8 (Lin, 2006) *A Borel measurable function* $f(t,x)$ *from* $\mathcal{T} \times \mathbb{R}$ *into* $\mathbb{R}$ *is said to satisfy linear growth condition in* x *if there exists a constant* $c > 0$ *such that*

$$|f(t,x)| \leq c(1 + |x|) \qquad \forall\ t\in \mathcal{T}; \qquad x, y \in \mathbb{R}. \tag{2.2}$$

Definition 9 (Pham, 2009) *Let* $W = (W_t)_{t\in\mathcal{T}}$ *be a 1-dimensional Brownian motion on a filtered probability space* $(\Omega, \mathcal{F}, \mathbb{F}, P)$*. Then an Ito process(stochastic integral) is a process* $X = (X_t)_{t\in\mathcal{T}}$ *valued in* $\mathbb{R}$ *such that almost surely*

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s \qquad t \in \mathcal{T}. \tag{2.3}$$

where X_0 *is* $\mathcal{F}_0$*-measurable,* $b : \mathcal{T} \times \mathbb{R} \to \mathbb{R}$ *and* $\sigma : \mathcal{T} \times \mathbb{R} \to \mathbb{R}$ *are progressively measurable processes valued in* $\mathbb{R}$ *such that*

$$\int_0^t |b_s|\,ds + \int_0^t |\sigma_s|^2\,ds < \infty \qquad a.s\ \forall\ t \in \mathcal{T}. \tag{2.4}$$

In differential form, Ito process can be written as

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \tag{2.5}$$

whereby $b(t,x)$ and $\sigma(t,x)$ are the drift and diffusion(dispersion) coefficients respectively.

The original motivation of Ito to introduce and develop the theory of stochastic integration was the construction of diffusion process by means of solving stochastic differential equations(SDEs) of the form (2.5). At this point we would like to introduce the notion of strong solution as well as Lipschitz and Linear growth conditions as evidence of the existence of the solution of (2.5).

Definition 10 (Strong solution. Karatzas *et al*, 1988) *A strong solution of SDE (2.5) on the given probability space $(\Omega, \mathcal{F}, P)$ and w.r.t the fixed Brownian motion $W = \{W_t; \mathcal{F}_t, t \in \mathcal{T}\}$ and initial condition Z is a process $X = \{X_t; t \in \mathcal{T}\}$ with continuous sample paths and with the following properties:*

i *X is adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ i.e X_t is $\mathcal{F}_t$-measurable.*

ii *$X_0 = Z$ almost surely.*

iii

$$P\left[\int_0^t \{|b(s, X_s)| + |\sigma(s, X_s)|^2\}ds < \infty\right] = 1 \qquad \forall t \in \mathcal{T} \tag{2.6}$$

iv *The integral version (2.3) of (2.5) holds almost surely.*

Theorem 3 (Existence and uniqueness theorem for SDEs. Øksendal, 2003) *Let $T > 0$ and $b(.,.) : [0,T] \times \mathbb{R} \to \mathbb{R}, \sigma(.,.) : [0,T] \times \mathbb{R} \to \mathbb{R}$ be Borel measurable functions satisfying conditions (2.1) and (2.2). On some probability space $(\Omega, \mathcal{F}, P)$, let Z be a random variable with values in $\mathbb{R}$ which is independent of the σ-algebra $\mathcal{F}_\infty^{\mathcal{W}}$ generated by the Brownian motion $W = \{W_t, t \in \mathcal{T}\}$ and such that*

$$\mathbb{E}\left[|Z|^2\right] < \infty \tag{2.7}$$

Then, the SDE (2.5) has a unique strong t-continuous solution $X_t(\omega)$ relative to W and the initial condition Z. The solution $X = \{X_t(\omega), t \in \mathcal{T}, \omega \in \Omega\}$ is *adapted to the filtration* $\mathcal{F}_t^Z = \sigma\{Z, W_s; 0 \leq s \leq t, t \in \mathcal{T}\}$ *and*

$$\mathbb{E}\left[\int_0^T |X_t|^2\, dt\right] < \infty. \tag{2.8}$$

For readers interested with the proof of this theorem should consult Øksendal(2003)

Theorem 4 (1-dimensional Ito formula. Pham, 2009) *Let* X_t *be an Ito process given by (2.5) and let* $g(t,x) \in \mathcal{C}^{1,2}([0,\infty) \times \mathbb{R})$, *i.e.* g *is twice continuously differentiable on* $[0,\infty) \times \mathbb{R}$. *Then for* $Y_t = g(t, X_t)$ *is again an Ito process, and*

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)dt + \frac{\partial g}{\partial x}(t, X_t)dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)dX_t.dX_t \tag{2.9}$$

where $dX_t.dX_t$ *is computed according to the rules* $dt.dt = dt.dW_t = dW_t = 0$ *and* $dW_t.dW_t = dt$.

Its proof is in Øksendal(2003)

2.1.2 Ito-diffusion and Generator

In this section, we introduce some properties of Ito-diffusion and the link between the second order partial differential operator and Ito diffusion that are most relevant for our work.

Definition 11 (Ndounkeu, 2010) *A control process is a progressively measurable process* $\theta = (\theta_t)_{t\in\mathcal{T}}$ *with values in* $\mathbb{U} \subseteq \mathbb{R}^n$, *i.e. the process* $\theta : \mathcal{T} \times \Omega \to \mathbb{U}$ *and* $(t,\omega) \mapsto \theta_t(\omega)$ *is* $\mathcal{B}_{\mathcal{T}} \times \mathcal{F}$*-measurable, where* $\mathcal{B}_{\mathcal{T}}$ *denotes the Borel* σ*-field of* $\mathcal{T}$.

Definition 12 (Ndounkeu, 2010) *The controlled process is the n-dimensional process solution of the following stochastic differential equation(SDE)*

$$dX_t^\theta = b(t, X_t, \theta_t)dt + \sigma(t, X_t, \theta_t)dW_t; \qquad X_0 = x \tag{2.10}$$

where $X_t \in \mathbb{R}^n$, $b : \mathcal{T} \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^n$ *and* $\sigma : \mathcal{T} \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}^{n\times m}$, W_t *is m-dimensional Brownian motion.*

Definition 13 (Ito diffusion. Ndounkeu, 2010) *A unique strong solution of (2.10) is called an Ito diffusion.*

We would like now to define the second order partial differential operator $\mathcal{L}$ as a generator of the Ito diffusion.

Definition 14 *Let $(X_t)_{t\geq 0}$ be an Ito diffusion in $\mathbb{R}^n$. The (infinitesimal) generator $\mathcal{L}$ of X_t is defined by:*

$$\mathcal{L}g(s,x) = \lim_{t\to s} \frac{\mathbb{E}^{s,x}[g(t,X_t)] - g(s,x)}{t-s} \qquad x \in \mathbb{R}^n \tag{2.11}$$

and g is in the domain $\mathcal{D}_{\mathcal{L}}$ which is the class of functions $g : \mathcal{T} \times \mathbb{R}^n \to \mathbb{R}$ for which the limit exists for all s, x.

Proposition 1 (Ndounkeu, 2010) *Let $\mathscr{L}$ be the differential operator defined on $\mathcal{C}^{1,2}$ by*

$$\mathscr{L} = \frac{\partial}{\partial t} + \sum_{i}^{n} b_i \frac{\partial}{\partial x} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} \tag{2.12}$$

with $(a_{ij} = (\sigma\sigma^T)_{ij})$ and let $g \in \mathcal{C}^{1,2}$ such that $\forall\ 0 \leq s \leq t,\ x \in \mathbb{R}$

$$\mathbb{E}^{s,x}\left[\int_s^t |\mathscr{L}g(u,X_u)|\, du\right] < \infty \quad \text{and} \quad \mathbb{E}^{s,x}\left[\int_s^t [(D_x g(u,X_u)^T \sigma(u,X_u)]^2 du\right] < \infty \tag{2.13}$$

Then $g \in \mathcal{D}_{\mathcal{L}}$ and $\mathcal{L}g = \mathscr{L}g$

The proof is in Ndounkeu (2010)

Remark 1 *Naturally, $\mathscr{L}$ is also a generator of Ito diffusion. Thus, when this operator $\mathscr{L}$ is applied to the function $g \in \mathcal{C}^{1,2}([0,\infty) \times \mathbb{R}^n, \mathbb{R})$, it leads to $\mathscr{L}g(t,x) = g_t(t,x) + (D_x g(t,x))^T b + \frac{1}{2} tr((D_{xx} g(t,x))a)$*
Here $D_x g$ and $D_{xx} g$ are the gradient vector and hessian matrix respectively of the function $x \to g(t,x) \in \mathcal{C}^{1,2}$, while $tr(A) = \sum_i^n a_{ii}$ is the trace of any square matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}, 1 \leq i,j \leq n$.

Theorem 5 (Dynkin's formula) *Let $g \in \mathcal{C}^{1,2}(\mathcal{T} \times \mathbb{R}^n, \mathbb{R})$. Suppose that τ is a stopping time satisfying $\mathbb{E}^x[\tau] < \infty$. Then*

$$\mathbb{E}^x[g(\tau, X_\tau)] = g(x) + \mathbb{E}^x\left[\int_0^\tau \mathscr{L}g(s,X_s)ds\right]. \tag{2.14}$$

The proof is in Øksendal (2003). If τ is the first exit-time from the bounded set $\mathcal{G} \in \mathbb{R}^n$, then Dynkin's formula holds for any $g \in \mathcal{C}^{1,2}$.

2.1.3 Strong formulation of the stochastic optimal control problem

In this section, we briefly discus some important elements which enable the strong formulation of an optimal control problem, so for more details one should consult Yong *et al*(1999) which is our main reference here. Given a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ satisfying the usual conditions on which an m-dimensional standard Brownian motion $W(.)$ is defined, consider the following controlled standard stochastic differential equation (2.10). From Definition 11, $\theta(t) \in \mathbb{U} \subseteq \mathbb{R}^d$ is a control process whose values can be chosen from the Borel set
$\mathcal{U}(\mathcal{T}) := \{\theta : \mathcal{T} \times \Omega \to \mathbb{U} | \theta(.) \text{ is } \{\mathcal{F}_t\}_{t\geq 0} - \text{adapted}\}$
at any instant t in order to control the process X_t and $\mathcal{T} = [0, T]$ for which T is fixed.

Thus the function $\theta(t)$ is representing the action, decision, or policy of the agents and investors. At any time instant the controller is knowledgeable about some information as specified by the information field, $\{\mathcal{F}_t\}_{t\geq 0}$ of what has happened up to that moment, but not able to foretell what is going to happen afterwards due to the uncertainty of the system(as a consequence, for any t the controller cannot exercise his/her decision $\theta(t)$ before the time t really comes). This non-anticipative restriction in mathematical terms can be represented as $\theta(t) = \theta(t, \omega)$ is a stochastic process and is $\mathcal{F}_t$-measurable i.e "$\theta(.)$ is $\{\mathcal{F}_t\}_{t\geq 0}$-adapted". Any $\theta(.) \in \mathcal{U}(\mathcal{T})$ is called a feasible control(Yong *et al*(1999)).

Now, we let $f : \mathcal{T} \times \mathbb{R}^n \times \mathbb{U} \to \mathbb{R}$(the utility rate or profit rate function) and $h : \mathcal{T} \times \mathbb{R}^n \to \mathbb{R}$(the bequest function) be given continuous functions, and let $\mathcal{G}$(the solvency set) be a fixed domain in $\mathcal{T} \times \mathbb{R}^n$ and let $\tau_{\mathcal{G}}$ be the first exit time after time $t = 0$ from $\mathcal{G}$ for the process $\{X_t^x\}_{t\geq 0}$, i.e. $\tau_{\mathcal{G}} := \inf\{t > 0; X_t \notin \mathcal{G}\}$.

We then suppose that

$$\mathbb{E}\left[\int_0^{\tau_{\mathcal{G}}} |f^{\theta(s)}(s, X_s)|ds + |h(\tau_{\mathcal{G}}, X_{\tau_{\mathcal{G}}})|\mathbb{I}_{\{\tau_{\mathcal{G}}<\infty\}}\right] < \infty \qquad \forall\ s, x, \theta \tag{2.15}$$

where $f^\theta(s, x) = f(s, x, \theta)$. Thus, we now define performance criterion(reward functional) measuring the performance of the controls as follows:

$$J^\theta(x) = \mathbb{E}\left[\int_0^{\tau_{\mathcal{G}}} f^{\theta(t)}(t, X_t, \theta_t)dt + h(X_{\tau_{\mathcal{G}}})\mathbb{I}_{\{\tau_{\mathcal{G}}<\infty\}}\right] \tag{2.16}$$

Definition 15 (Yong *et al*, 1999) *Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be given filtered probability space satisfying the usual conditions and let $W(.)$ be a given m-dimensional standard $\mathcal{F}_t$-Brownian motion. A control $\theta(.)$ is called admissible control and $(X(.), \theta(.))$ an admissible pair, if*

i. $\theta(.) \in \mathcal{U}(\mathcal{T})$

ii. X(.) is a unique solution of the equation (2.15) in the sense of Theorem 2.

iii. $f(., X(.), \theta(.)) \in \mathscr{L}^1_{\mathcal{F}}(\mathcal{T}, \mathbb{R})$ *and* $h(X_{\tau_{\mathcal{G}}}) \in \mathscr{L}_{\mathcal{F}_{\tau_{\mathcal{G}}}}(\Omega, \mathbb{R})$.

iv. The state constraints are satisfied (if specified).

The set of all admissible controls is denoted by $\mathcal{U}(\mathcal{T})$. Our stochastic optimal control problem under strong formulation can then be stated as follows:

$$V(x) := \sup_{\theta(t,\omega)}\{J^\theta(x)\}. \tag{2.17}$$

So, the problem is for each $x \in \mathcal{G}$, we need to find the control
$\theta^* = \theta^*(t, x, \omega) \in \mathcal{U}(\mathcal{T})$ such that $V(x) = J^{\theta^*}(x)$, where the supremum is taken over a given family $\mathcal{U}(\mathcal{T})$ of all admissible controls, contained in the set of all $\mathcal{F}_t$-adapted process $\theta(t)$ with values in $\mathbb{U}$(Yong *et al*(1999)).
Such a control θ^* if it exists is called an *optimal control* and V is called the *optimal performance (value function)*. So in the next chapter, we shall use these ideas to the real problem we are going to formulate.

2.1.4 Bellman's Principle of optimality and Hamilton-Jacobi-Bellman (HJB)equation

Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a given filtered probability space satisfying the usual conditions, on which a standard m-dimensional Brownian motion W(.) is defined. Consider the stochastic controlled system (2.10) with the performance functional (2.16) and the admissible control $\mathcal{U}^s(\mathcal{T})$ in Definition 14 where the superscript 's'indicates that the strong formulation is being considered(Yong *et al*(1999)). For any $\theta(.) \in \mathcal{U}^s(\mathcal{T})$ equation (2.10) admits a unique strong solution, and the performance function (2.16) is well-defined. Now let $T > 0$ be given and let $U \in \mathbb{R}^n$. For any $(t_0, y) \in [0, T] \times \mathbb{R}^n$, we consider the modified state equation

$$\begin{cases} dX_t^\theta = b(t, X_t, \theta_t)dt + \sigma(t, X_t, \theta_t)dW_t, & t \in [0, T] \\ X_{t_0} = y \qquad y \in \mathbb{R}^n. \end{cases} \tag{2.18}$$

along with the performance functional

$$J(t_0, x, \theta) = \mathbb{E}\left[\int_{t_0}^{T} f(t, X_t, \theta_t)dt + h(X_T)\right] \tag{2.19}$$

where θ satisfies the conditions of Definition 15 and f and h are uniformly continuous satisfying (2.15) and b and σ satisfy the conditions (2.1) and (2.2).
We now state the stochastic version of Bellman's principle of optimality, which is commonly known as the Dynamic programming principle(DPP).

Theorem 6 (Bellman's equation. SaB, 2007) *For all* $(t_0, y) \in [0, T] \times \mathbb{R}^n$ *and* $t_1 \in [t_0, T]$

$$V(t_0, y) = \sup_{\theta \in \mathcal{U}} \mathbb{E}^{t_0, y}\left[\int_{t_0}^{t_1} \phi(s, X_s, \theta_s)ds + V(t_1, X_{t_1})\right] \qquad \forall\ 0 \leq t_0 \leq t_1 \leq T. \tag{2.20}$$

Briefly the principle says that, an optimal policy from t_0 to T passing through t_1 is also optimal in $[t_1, T]$. Its thorough proof is in SaB(2007), and one can also find it in Ndounkeu(2010) and Yong *et al.*(1999).

Under certain assumption on the value function, Bellman's equation, can be used to derive the so called Hamilton-Jacobi-Bellman equation. With the application of Ito's formula (2.7) to the value function V if it is smooth enough, and with some reasoning lead to HJB equation.

Proposition 2 (Yong *et al*, 1999) *Suppose that the conditions (2.1), (2.2) and (2.15) hold and V is smooth enough(ie $V \in \mathcal{C}^{\infty,\in}([0,T] \times \mathbb{R}^n, \mathbb{R})$). Then the value function V is the solution of a second order non-linear partial differential equation*

$$\sup_{\theta \in \mathcal{U}} \{\phi(t,x,\theta) + V_t(t,x) + (D_x V(t,x))^T b(t,x,\theta) + \frac{1}{2} tr((D_{xx} V(t,x)) a(t,x,\theta))\} = 0. \tag{2.21}$$

Rewriting equation (2.21) using differential operator in (2.12) satisfying (2.13) depending on θ, eventually we obtain

$$\sup_{\theta \in \mathcal{U}} \{\phi(t,x,\theta) + \mathscr{L}^\theta V(t,x)\} = 0 \tag{2.22}$$

where the supremum is taken over all the admissible control. So, for a fixed x, the quantity will be maximized only through $\theta(t,x) \in \mathbb{U}$. Therefore the supremum will now be taken over $\mathbb{U}$ and the above equation is then written as

$$\begin{cases} \sup_{\theta \in \mathbb{U}} \{\mathscr{L}^\theta V(t,x)\} = 0 \\ V(T,x) = U(T,x) = U(x). \end{cases} \tag{2.23}$$

which is then called HJB equation, and also known as the dynamic programming equation, short DPE and $U(x)$ being utility function. It is a non-linear PDEs which sometimes can be very complicated and hard to get solved. However, Bellman's principle of optimality and HJB theory would be our right and useful tools in solving the stochastic control problem we are soon going to introduce in the next chapter.

2.2 Literature review

The problem of portfolio optimization is a well studied problem in mathematical finance and is considered to be a stochastic control problem. In most papers, researchers and academicians are highly looking after explicit solution to the problem, e.g Merton(1969, 1970, 1971, 1976), Øksendal(2003), Korn(2008), Korn *et al.*(2001, 2002), Luo *et al.*(2011), Benth *et al.*(2003). A major step in the direction of quantitative management of portfolios was made by Harry Markowitz in his paper "*Portfolio selection*" published in 1952 in the Journal of finance(Fabozzi *et al.*, 2006). The idea introduced in this aticle have come to build the foundations of what is now popularly referred to as *Mean-variance analysis, Mean-variance optimization and Modern portfolio theory(MPT).*

Merton(1976) examined the combined problem of optimal portfolio selection and consumption rules for an individual in a continuous time model where his income is generated by the returns on assets and the returns are stochastic. He derived the optimality equations for a multi-assets problem when the rate of returns are generated by a Wiener Brownian-motion. The two asset model with constant relative risk-aversion and the constant absolute risk-aversion were examined in detail to obtain explicit solution. The very recent paper by Luo *et al.*(2011) concerned with one kind of optimal control problem of portfolio and consumption choice in a financial market, where an investor has three investment options, namely, bond, stock and the foreign exchange deposit. They used dynamic programming principle(DPP) and Hamilton-Jacobi-Bellman's(HJB) equation as mathematical tools to obtain the explicit optimal portfolio and consumption choice for the power utility function case.

Since the seminal work of Merton(1969), who explored stochastic optimal control techniques to provide a closed form solution to the problem, a large volume of research has been done to extend Merton's paradigm to other frameworks and portfolio optimization problems. For instance Campbell *et al.*(2001), Fleming *et al.*(2004), Korn(2008), Korn *et al.*(2001, 2009), Øksendal(2003), Pang(2006, 2009), Platen *et al.*(2007), Seierstad(2009).

Benth *et al.*(2003) studied the Merton's classical portfolio optimization problem for an investor who can trade in a risk-free bond and a stock. In their study, they included stochastic volatility in the dynamics of the risky asset and used model driven by a superposition of Non-Gaussian Ornstein-Uhlenbeck process. They also employed the DPP method to derive an explicitly trading strategies and value function expression via Feynmann Kaç formula and verified with power utility. In the book Øksendal(2003), the problem of linear stochastic regulator and an optimal portfolio selection were discussed. He precisely showed how HJB equations provide a nice solutions to the stochastic control problem in the case where only Markov controls are considered and when arbitrary $\mathcal{F}_t^{(m)}$-adapted controls are used with conditions imposed. He elaborates how to formulate performance criterion and then transformation to PDE by the use of Ito diffusion(or differential operator). The HJB equation is applied to form a non-linear PDE which he then solved by careful selection of suitable utility functions. Their results confirm that it is optimal to invest according to Merton's strategy, however, one updates the investment according to the level of volatility.

Haugh *et al.*(2006), analysed dynamic portfolio choice problems using an approximate dynamic programming(ADP) algorithm. They extended the algorithm to the case of constraints on borrowing and implemented the duality-based simulation procedure for estimating bounds on the true value function. Through examples, they demonstrated that the ADP solutions exhibits a high degree of accuracy which indicates that the method is a promising approach to tackling challenging practical problems in the area of asset allocation and portfolio choice. The very current study by Øksendal *et al.*(2011) considered the case where optimal portfolio problems for the markets are modelled by (possibly non-Markovian) jump diffusions. They described mathematically the situation as a stochastic differential game, where one of the players(the agent) is trying to find the portfolio which maximizes the utility of the terminal wealth, while the second player (the market) is controlling some of the unknown parameters of the market(eg the underlying probability measure, representing a model uncertainty problem) and trying to minimize this maximal utility of the agent. The problem is approached by transforming it to a stochastic differential game for backward differential equations(BSDE games). Using comparison theorems for BSEDs with jumps, they arrived at a criteria for the solution of such games, in the form of a kind of Non-Markovian analogue of the HJBI equation.

Fleming *et al.*(2004) considered a stochastic control model in which an economic unit has productive capital and also liabilities in the form of debt. The worth of capital changes overtime through investment and also through random Brownian fluctuations in the unit price of capital. Income from production is also subject to random Brownian fluctuations. The dynamic programming method was used to obtain the optimal control policies. IIhan *et al.*(2004) studied the problem of portfolio optimization in an incomplete market using derivatives as well as basic asset such as stocks. They discussed the computational tractability obtained by assuming exponential utility and the connection to the method of utility-indifference pricing. In particular, they showed that optimal number of derivatives to invest in is given by the optimizer in the Legendre transform of the indifference price as a function of quantity evaluated at the market price. They suggested some asymptotic approximations for the optimal derivative holding, first when it might be small, and second, in the case of slowly varying volatility. Jurek *et al.*(2007) derived an optimal dynamic strategy for arbitrageurs with a finite horizon and non-myopic preferences facing a mean-reverting arbitrage opportunity. They showed analytically that there is a critical level of mispricing beyond which further divergence precipitates a reduction in the allocation as the arbitrageurs typically bet against mispricing.

Dmitrasinovic-Vidovic *et al.*(2011) investigated portfolios consisting of instruments whose logarithms are mean-reverting. They derived an analytic expression for the expected wealth and the quantile-based risk-measure capital at risk under the assumption that portfolios are constant. They assumed that short-selling and borrowing are allowed, then they solved the problems of global minimum capital at risk and the problem of finding maximal wealth subject to constrained capital at risk. They finally provided some numerical examples which illustrated that the presence and strength of mean-reversion in an asset model have a significant effect on optimal portfolio management. Munk *et al.*(2004), in their paper, considered the optimal asset allocation strategy of a power utility investor who can invest in cash(a bank accout), nominal bonds and stock in a model that exhibits mean-reverting stock returns and real interest rate uncertainty. They calibrated the capital market to US stock, bond and inflation data. They illustrated an optimal asset allocation by performing a calibration exercise where risk aversion parameters and time horizon are fitted.

CHAPTER THREE
FORMULATION OF THE OPTIMAL PORTFOLIO PROBLEM

3.1 Formulation of the Wealth SDE

In this chapter we are going to formulate the stochastic portfolio optimization problem in continuous time and use the stochastic control techniques to find the optimal portfolio value by maximizing the utility of the wealth at some future time T.

Throughout this work, we shall be considering the dynamic system characterized by its state at any time, and evolving in an environment formalized by a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$ for $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$ satisfying the usual condition on which a 1-dimensional standard Brownian motion $W = \{W_t, t \geq 0\}$ valued in $\Re$ is defined(Pham(2009)).

We present the problem of portfolio allocation by considering the Black-Scholes financial market with two investment possibilities namely: a risk free asset with positive price evolving as

$$dB_t = rB_t dt \qquad B_0 = 1. \tag{3.1}$$

and a risky asset with price at time t described dynamically by the geometric mean-reversion model

$$\begin{cases} dS_t = \kappa(\mu - \ln S_t)S_t dt + \sigma S_t dW_t \\ S_0 = s_0. \end{cases} \tag{3.2}$$

Where the parameters of the market κ, σ, μ are positive constant such that μ represents the long term mean equilibrium(i.e the value around which the future trajectories will converge in a long run), κ is the speed of that convergence and σ is the degree of volatility.

Thus if the incremental change in the stock price is governed by the above geometric mean reversion relation then, by solving (3.2) we obtain the price of the stock at time t given hereunder (assuming it, a unique solution) as

$$S_t = \exp\left(e^{-\kappa t}\ln s_0 + \left(\mu - \frac{\sigma^2}{2\kappa}\right)(1 - e^{-\kappa t}) + \int_0^t \sigma e^{-\kappa(t-u)} dW_u\right). \tag{3.3}$$

We now consider the investment problem of an investor who has access to the capital market and wants to transfer current wealth $X_0 = x_0$ into the bond and stock. His/her preference is to choose a dynamic portfolio strategy in order to maximize the expected utility of wealth at some future time T. Thus, to describe the investor's actions, we introduce the portfolio strategies.

Definition 16 (Ndounkeu, 2010) : *Portfolio strategy is a two dimensional stochastic process*

$$\pi = \{\pi(t) = (\Pi_0(t), \Pi(t)), t \in [0, T]\} \tag{3.4}$$

satisfying the following conditions

1 π is progressively measurable.

2 π is adapted i.e $\forall_t$, $\pi(t)$ is $\mathcal{F}_t$-measurable.

The financial interpretation of the portfolio strategy is that $\Pi_0(t)$ is the number of units of bonds held by the investor at time t and $\Pi(t)$ is the number of units of stocks held by the investor at time t.

Therefore the wealth (portfolio value) $(X_t)_{t\in[0,T]}$ of an investor with initial capital $x_0 > 0$ is such that

$$X_t = \Pi_0(t)B_t + \Pi(t)S_t. \tag{3.5}$$

We regard this pair $(\Pi_0(t), \Pi(t))$ as self-financing if the corresponding wealth process $(X_t)_t$ is a continuous and adapted process such that

$$X_t = x_0 + \int_0^t \Pi_0(u)dB_u + \int_0^t \Pi(u)dS_u. \tag{3.6}$$

This implies that changes in the wealth are only due to changes in the bond or stock prices, i.e. no external inflow or outflow of cash.

The investor needs to monitor his/her wealth, and therefore we can set the fraction θ_t of the wealth invested in stocks to be the control of the system at time t. Thus we have

$$\Pi(t) = \frac{\theta_t X_t}{S_t} \qquad \Pi_0(t) = \frac{(1-\theta_t)X_t}{B_t} \tag{3.7}$$

We assume $\theta(t)$ to be almost surely continuous in $t \in [0,T]$ and since π is assumed to be self-financing, then by (3.6) we have

$$dX_t = \Pi_0(t)dB_t + \Pi(t)dS_t. \tag{3.8}$$

and by (3.1) and (3.2) we can write (3.8) in the the following form

$$dX_t = \Pi_0(t)rB_t dt + \Pi(t)[\kappa(\mu - \ln S_t)S_t dt + \sigma S_t dW_t]$$

and then through collection of like terms we obtain

$$= [r\Pi_0(t)B_t + \kappa(\mu - \ln S_t)\Pi(t)S_t]\, dt + \sigma\Pi(t)S_t\, dW_t.$$

With further elimination of S_t and B_t by the help of (3.7) we finally obtain

$$dX_t = \left[(1-\theta_t)r + \kappa\left(\mu - \ln\left(\frac{\theta_t X_t}{\Pi(t)}\right)\right)\theta_t\right] X_t dt + \sigma\theta_t X_t dW_t. \tag{3.9}$$

We would like to make equation (3.9) look much more beautiful, then we let $Y_t = \mu - \ln\left(\frac{\theta_t X_t}{\Pi(t)}\right)$ and substitute it into (3.9) to get

$$dX_t = [r + (\kappa Y_t - r)\theta_t]X_t dt + \sigma\theta_t X_t dW_t \tag{3.10}$$

which is the stochastic differential equation for the wealth. We logically assume that the investor has complete information from the market at all instant, i.e. $\theta(t)$ is adapted. Therefore the investment policy is defined by an $\mathbb{F}$-adapted process $\{\theta = (\theta_t), t \in [0,T]\}$ which is a control process. In this case given a portfolio process θ, plausibly sounds convenient to rewrite (3.10) as

$$\begin{cases} dX^{\theta}_{x,t} = [r + (\kappa Y_t - r)\theta_t]X_t dt + \sigma\theta_t X_t dW_t \quad t \in [0,T] \\ X_0 = x \qquad x \in \mathbb{R}. \end{cases} \tag{3.11}$$

The notation X_x^θ is used to emphasize the dependence of the wealth process on the initial wealth and the control. If the above equation (3.11) has a unique solution X, for a given data, then X is called the controlled process, as his dynamics driven by the actions of the control process θ.

3.2 Utility functions

After having stated the process modelling the wealth of the investor, we would like to make a brief survey on the utility functions. In the investment process, the investor is faced with the problem of making decisions under risk. Therefore, it is necessary to associate each of the decisions with preference, that is the measure of degree of satisfaction of the investor. The classical modelling for describing the behaviour and preferences of the agents and investors are *the expected utility criterion* and *mean-variance criterion*.

By considering the first criterion, Von Neumann and Morgenstern suggested that, the preferences can be represented through the expectation of some functions, called *utility functions*(Pham(2009)).

Definition 17 (Khati, 2011) *: Utility function is a function* $U : [0, \infty) \to [0, \infty)$ *such that*

1. $U(x)$ *is twice differentiable i.e* $U \in \mathcal{C}^2(\mathbb{R})$.
2. $U(x)$ *is strictly increasing* $(U'(x) > 0)$*; Non-satiation.*
3. $U(x)$ *is strictly concave* $(U''(x) < 0)$*; Risk-averse*
4. $U(x) > 0 \ \forall x \in \mathbb{R}_+$ *and* $U(x) = -\infty \ \forall x \in \mathbb{R}_-$

where x represents the amount of the investor's wealth.

According to Arrow(1965) and Pratt(1964), there are two measures of degree of risk-aversion obtained from the utility functions, namely(Ndounkeu, 2010):

- Absolute risk aversion(ARA)function the ratio $A(x) = -\frac{U''(x)}{U'(x)}$
- Relative risk aversion(RRA)function the ration $R(x) = -x\frac{U''(x)}{U'(x)}$

Investors may use the above measures to make decisions in the process of optimizing the portfolio while taking into account the risk associated with each asset. Basically, the utility functions allow us to visualize the preference relations among different levels of wealth and different portfolio strategies. We would like to give some examples of the utility functions that are frequently used in financial models and the intuition behind risk aversion measures.

Example 1 (Exponential utility) $U(x) = 1 - e^{-ax}, \qquad a > 0$
The ARA is $A(x) = a$, which is constant, meaning that the behaviour of the investor towards the risk does not depend on the initial wealth.

Example 2 (Logarithmic utility) $U(x) = \log x$
The ARA is $A(x) = \frac{1}{x}$, which decreases as the wealth increases. This means that, someone with higher capital is less afraid of taking risk than someone with lower wealth, which indeed makes sense from the economic point of view.

Example 3 (Power utility) $U(x) = \frac{x^\alpha}{\alpha}, \qquad 0 < \alpha < 1$
The ARA is $A(x) = \frac{1-\alpha}{x}$, which is a decreasing function of wealth. This implies that, the investor with higher capital is less afraid of taking risk than someone with lower capital, which is indeed much more meaningful in economy sense.

The investor's goal is to find the process $(\theta_t)_t$ that enables him/her to make as much money as possible. Thus in that regard we assign the utility function U to wealth so that we maximize the expected utility of the wealth at future time.

3.3 The stochastic optimal control problem

From(3.11), we suppose that $X_{t_0} = x > 0$ at time t_0. The investor wants to maximize the expected utility of the wealth at some future time $t_1 > t_0$. We assume that $0 \leq \theta_t \leq 1$, and by the concept of utility function from which we have assigned the utility function U to the wealth, then we begin by defining the *Optimization criterion* or a *Reward function* as

$$J^{\theta}(t_0, x) = \mathbb{E}^{t_0,x}[U(X^{\theta}_{\tau_{\mathcal{G}}})|X^{\theta}_{t_0} = x] \tag{3.12}$$

where $\tau_{\mathcal{G}}$ is the first exit time from the region $\mathcal{G} = \{(s, x_1); s < t_1, 0 < x < x_1\}$ defined as $\tau_{\mathcal{G}} = \inf\{t_1 > t_0; (t_1, X_{t_1}) \notin \mathcal{G}\} \leq \infty$ (Øksendal(2003), and Theorem 2(chapter 2). Actually, x_1 is the amount of the wealth at any time $s < t_1$ before exit from the region $\mathcal{G}$.

This is a performance criterion of the form (2.16) in chapter 2, with $f = 0$ and $h = U$.

We need to maximize the expected utility of the wealth $\mathbb{E}[U(X^{\theta})]$ over the class $\mathcal{U}(t, x)$ of all admissible portfolio strategies θ that satisfy

$$\mathbb{E}[U(X^{\theta}_{\tau_{\mathcal{G}}})] < \infty \tag{3.13}$$

Eventually we define the *value function* of the control problem which is actually our *stochastic optimal control problem* as follows

$$V(t_0, x) = \sup_{\theta \in \mathcal{U}(t_0,x)} \left\{ J^{\theta}(t_0, x); 0 \leq \theta \leq 1, \quad \text{where} \quad \theta \quad \text{is Markov control} \right\} \tag{3.14}$$

We wish to find an optimal strategy θ^* for which an optimal value $V(t_0, x)$ is attained, i.e. $V(t_0, x) = J(t_0, x, \theta^*)$.

3.4 Dynamic programming and Hamilton-Jacobi-Bellman equation

At this juncture, we would like to solve the stochastic optimal control problem (3.14) by maximizing the performance function (3.12) satisfying condition (3.13) and subject to the state(wealth) equation (3.11). We can see that, an optimal control problem (3.14) is similar to Bellman's equation (2.20) in Theorem 6, with $\phi = 0$. We apply differential operator $\mathscr{L}^{\theta}$ to the value function V in (3.14) to get

$$\mathscr{L}^{\theta} V(t, x) = V_t(t, x) + b(t, x, \theta) V_x(t, x) + \frac{1}{2}\sigma^2(t, x, \theta) V_{xx}(t, x) \tag{3.15}$$

whereby, from the wealth SDE (3.11), we have

$$b(t,x,\theta) = (r + (\kappa y - r)\theta)x \quad \text{and} \quad \sigma(t,x,\theta) = \sigma\theta x$$

and $y = \mu - \ln\left(\frac{\theta x}{\Pi}\right)$, being the substitution made in (3.9) to yield (3.11). Hence from equation (3.15), we deduce the HJB equation

$$\begin{cases} \sup_{\theta\in\mathcal{U}}\{\mathscr{L}^{\theta}V(t,x)\} = 0 \quad \text{for} \quad (t,x)\in\mathcal{G} \\ V(t,x) = U(x) \quad \text{for} \quad t = t_1 \\ V(t,0) = U(0) = 0 \quad \text{for} \quad t < t_1 \end{cases} \tag{3.16}$$

where $U(x)$ stands for any utility function that shall be applied in here.

Therefore, for all $(t,x)\in\mathcal{G}$, our main interest is to find the value $\theta = \theta(t,x)$ of which, in turn, it maximizes the function

$$\xi(\theta) = \mathscr{L}^{\theta}V(t,x) = \frac{\partial V}{\partial t} + x(r + (\kappa y - r)\theta)\frac{\partial V}{\partial x} + \frac{1}{2}\sigma^2\theta^2x^2\frac{\partial^2 V}{\partial x^2} \tag{3.17}$$

Since $y = \mu - \ln\left(\frac{\theta x}{\Pi}\right)$, we let $f(\lambda) = \ln\lambda$ such that $y = \mu - f(\lambda)$. Thus for simplicity, before dealing with the value of $\theta = \theta(t,x)$ which maximizes $\xi(\theta)$ above, we would like first to linearly approximate the function $f(\lambda) = \ln\lambda$ by Taylor series at $\lambda_0 = 1$. Thus by Taylor series we have

$$f(\lambda) = (\lambda - 1) - \frac{(\lambda-1)^2}{2} + \frac{(\lambda-1)^3}{3} + \ldots \approx \lambda - 1$$

Therefore, making substitution for λ, we get an approximated linear expression

$$f(\lambda) = f\left(\frac{\theta x}{\Pi}\right) \approx \frac{\theta x}{\Pi} - 1 \tag{3.18}$$

Plug equation (3.18) into the function $\xi(\theta)$ in equation (3.17), to get the approximated function $\xi(\theta)$ which we name it as $\eta(\theta)$.

$$\xi(\theta) \approx \eta(\theta) = V_t(t,x) + x\left[r(1-\theta) + \kappa\theta\left(\mu + 1 - \frac{\theta x}{\Pi}\right)\right]V_x(t,x) + \frac{1}{2}\sigma^2x^2\theta^2V_{xx}(t,x). \tag{3.19}$$

From equation (3.19), we can therefore modify the HJB equation (3.16) and become

$$\begin{cases} \sup_{\theta\in\mathcal{U}}\{\mathscr{L}^{\theta}V(t,x)\} \approx \sup_{\theta\in\mathcal{U}}\{\eta(\theta)\} = 0 \quad \text{for} \quad (t,x)\in\mathcal{G} \\ V(t,x) = U(x) \quad \text{for} \quad t = t_1 \\ V(t,0) = U(0) = 0 \quad \text{for} \quad t < t_1 \end{cases} \tag{3.20}$$

which is the same as

$$\begin{cases} \sup_{\theta\in\mathcal{U}}\{V_t(t,x) + x\left[r(1-\theta) + \kappa\theta\left(\mu + 1 - \frac{\theta x}{\Pi}\right)\right]V_x(t,x) + \frac{1}{2}\sigma^2x^2\theta^2V_{xx}(t,x)\} = 0 \\ \text{for} \quad (t,x)\in\mathcal{G} \\ V(t,x) = U(x) \quad \text{for} \quad t = t_1 \\ V(t,0) = U(0) = 0 \quad \text{for} \quad t < t_1. \end{cases} \tag{3.21}$$

Now, assuming that, V satisfies condition 2 and 3 of Definition 17, and that $\eta(\theta)$ has a maximum value at some $\theta(t,x)$, then we have

$$\frac{d\eta}{d\theta} = (\kappa(\mu+1) - r)\,xV_x - \frac{2\kappa\theta x^2}{\Pi}V_x + \sigma^2\theta x^2V_{xx} = 0$$

and solving for θ from the expression above we finally obtain

$$\theta = \theta(t,x) = \frac{(\kappa(\mu+1) - r)\,V_x}{\left(\frac{2\kappa V_x}{\Pi} - \sigma^2V_{xx}\right)} \tag{3.22}$$

By substituting equation (3.22) into HJB equation (3.21), we obtain the partial differential equation

$$\begin{cases} V_t + rxV_x + \frac{1}{2}\frac{(\kappa(\mu+1)-r)^2V_x^2}{\frac{2\kappa V_x}{\Pi} - \sigma^2V_{xx}} = 0 \quad \text{for} \quad t < t_1, x > 0 \\ V(t,x) = U(x) \quad \text{for} \quad t = t_1. \end{cases} \tag{3.23}$$

Which is a boundary value problem for V. This boundary value problem is extremely hard to solve for general utility function U. Thus, the work would be simplified if we consider the specific examples of utility functions given in section 3.2. We start to implement this by stating hereunder, the first theorem which thereafter will be followed by its proof.

Theorem 7 *Suppose that, for all $\mathbb{F}$-adapted control process $\theta \in (0,1)$ of the wealth $x > 0$, the solution for the boundary value problem (3.23) exists, and that, the investor's behaviour is modeled by the power utility function*

$$U(x) = \frac{x^\alpha}{\alpha}; \quad 0 < \alpha < 1.$$

Then, the optimal control strategy θ^* is given by

$$\theta^*(t,x) = \frac{\varepsilon}{\delta x + \varrho} \tag{3.24}$$

where the constants ε, δ and ϱ are positive and depend on the market parameters.

Proof 1

Since V is a function of two variables t and x, then by separation of variables (or product method), the goal is to have a solution of the form

$$V(t,x) = h(t)U(x) = h(t)\frac{x^\alpha}{\alpha} \tag{3.25}$$

satisfying the boundary value problem (3.23), and therefore, it is required to solve for h. From (3.25), it is found that

$$V_t = h'(t)\frac{x^\alpha}{\alpha}, \quad V_x = h(t)x^{\alpha-1} \text{ and } V_{xx} = h(t)(\alpha-1)x^{\alpha-2}. \tag{3.26}$$

Then, substituting equation (3.26) into BVP (3.23), the equation below is obtained

$$x^\alpha \left(h' + h\alpha \left(r + \frac{(\kappa(\mu+1) - r)^2}{\frac{2\kappa x}{\Pi} - \sigma^2(1-\alpha)} \right) \right) = 0 \text{ for } t < t_1.$$

Since $x^\alpha \neq 0$ as $x > 0$ then a simplified equation below is obtained

$$h' + \beta(x)h = 0 \tag{3.27}$$

which is a separable differential equation. The equation (3.27) is solved, while setting $h(t) = 1$, for $t = t_1$. The solution is then found to be

$$h(t) = e^{\beta(x)(t_1 - t)}. \tag{3.28}$$

Hence equation (3.25) becomes

$$V(t,x) = h(t)\frac{x^\alpha}{\alpha} = e^{\beta(x)(t_1-t)}\frac{x^\alpha}{\alpha} \tag{3.29}$$

where $\beta(x) = r\alpha + \frac{(\kappa(\mu+1)-r)^2\alpha}{\frac{2\kappa x}{\Pi}-\sigma^2(1-\alpha)}$. Now, from (3.29) the partial derivatives V_x and V_{xx} are obtained, which are then plugged into (3.22) to get

$$\theta^* = \theta^*(t,x) = \frac{\kappa(\mu+1)-r}{\frac{2\kappa x}{\Pi}+\sigma^2(1-\alpha)} \tag{3.30}$$

which is equivalent to (3.24) with $\varepsilon = \kappa(\mu+1)-r$, $\delta = 2\kappa/\Pi$ and $\varrho = \sigma^2(1-\alpha)$, and the proof is hence complete. □

And consequently, the equation (3.29) is then the solution of the HJB equation (3.21), provided that $\theta^* \in (0,1)$.

Theorem 8 *Suppose that the first hypothesis in Theorem 7 is considered, and suppose that, an exponential utility function.*

$$U(x) = -e^{-ax}, \qquad a > 0$$

is considered in the modeling of the investor's behaviour in the market playground. Then the optimal policy is inversely proportional to the wealth. That is

$$\theta^*(t,x) \propto \frac{1}{x}$$

Proof 2

By the separation technique, the proof begins by assuming that, the value function is given by $V(t,x) = h(t)U(x)$ such that:

$$V_t = -h'e^{-ax}, \qquad V_x = hae^{-ax} \text{ and } V_{xx} = -ha^2e^{-ax} \tag{3.31}$$

Therefore, by plugging (3.31) into (3.23) and then look forward to obtain the function h. By substituting equation (3.31) into (3.23), then result to

$$e^{-ax}\left[h' - ha\left(rx + \frac{1}{2}\frac{(\kappa(\mu+1)-r)^2}{\frac{2\kappa}{\Pi}+\sigma^2 a}\right)\right] = 0 \text{ for } t < t_1$$

Since $e^{-ax} \neq 0$ then, through setting $h(t) = 1$ for $t = t_1$, it appears to have separable ordinary differential equation

$$h' + \gamma(x)h = 0$$

from which the solution is simply obtained. That is

$$h(t) = e^{\gamma(x)(t_1 - t)} \tag{3.32}$$

whereby $\gamma(x) = -a\left(rx + \frac{1}{2}\frac{(\kappa(\mu+1)-r)^2}{\frac{2\kappa}{\Pi}+\sigma^2 a}\right)$. Hence, the solution $V(t,x)$ is achieved such that

$$V(t,x) = h(t)U(x) = -e^{\gamma(x)(t_1-t)}.e^{-ax}. \tag{3.33}$$

So, from (3.33), the partial derivatives

$$\begin{cases} V_x = -aV \\ V_{xx} = a^2 V \end{cases}$$

are easily obtained, and then substitution into (3.22) is performed to get another expression for the optimal strategy θ^* in the case of exponential utility considered as the investor's behavioral measure. That is

$$\theta^* = \theta^*(t,x) = \frac{(\kappa(\mu+1) - r)}{\left(\frac{2\kappa}{\Pi} + \sigma^2 a\right)x} \tag{3.34}$$

and the proof is complete. □

The optimal control obtained in both cases of utility functions, depends on the wealth x, the market parameters k, μ, r and σ as well as α for the first case and a for the second case. The results obtained here look different from the other results which have been found by other researchers.

The differences actually arise from the fact that, most of the researches which have been conducted particularly in the optimal portfolio problems, the dynamics of the risky assets(stocks) have been described by the geometric Brownian motion. The controlled SDE for the wealth process formulated from that model

leads to the value function from which the optimal policy is obtained and found to be independent from time and the wealth in particular.
While for our case the dynamics of the risky asset is described by the geometric mean reversion(GMR) processes as the equation (3.2) shows. Thus the formulation of the controlled wealth SDE incorporates the deterministic differential equation (3.1) and the GMR model (3.2), and from there the value function (3.14) is defined and hence extraction of optimal policies which depend on the wealth and the market parameters as stated above.

Remark 2

- *The stochastic control technique has been the fundamental tool in the formulation and solving the portfolio problem.*
- *The solution of this portfolio problem involves formulation of HJB equation which is a non-linear PDE of second order describing the local behaviour of the value function. And solving it, involves finding the value at which a function attains its maximum and then solve a PDE.*
- *To compute explicitly the value function and the optimal policy, we linearly approximated the logarithmic function of theta (theta is the control of the wealth) appeared in the HJB equation due to GMR model used to model the risky asset.*
- *The specific utility functions which are members of HARA(Hyperbolic Absolute Risk Aversion)family have been used to compute analytically and explicitly the value function and the optimal investment rules.*

CHAPTER FOUR
ANALYSIS OF THE RESULTS

4.1 Introduction

In this chapter we intend to make analysis of the results obtained in the previous chapter. We use MATLAB software to implement the simulation of the optimal strategy and study its behaviour in relation to the wealth. We also implement the simulation of the value function with respect to time and the wealth for the same market parameters used in the simulation of optimal policy. For both cases, power utility and exponential utility, the results are analyzed differently.

4.2 Simulation in the case of power utility

At this juncture, we implement the simulation of the results obtained by solving the portfolio problem when the power utility used as the measure of the investor's behaviour.

4.2.1 Simulation of optimal strategy

We implement the simulation of the optimal strategy with respect to wealth of the portfolio with the market parameters $\mu = 10.54, \kappa = 0.3, \sigma = 0.97, r = 0.05, \Pi = 2,$ and $\alpha = 0.5$. The figure 4.1 shows that, the optimal investment strategy decreases almost to zero as the wealth increases. This implies that, as the investor becomes richer the less he invests in risky assets. This result looks somewhat absurd as it contradicts with the economic interpretation of the absolute risk aversion(ARA) in example 3 which signifies that, someone with higher capital is less afraid of taking risk in investing on risky assets. On the other hand, the result concurs exactly with what is happening in real life situation, whereby as someone gains more wealth, then deposits most of his/her wealth in bank accounts than investing in risky assets.

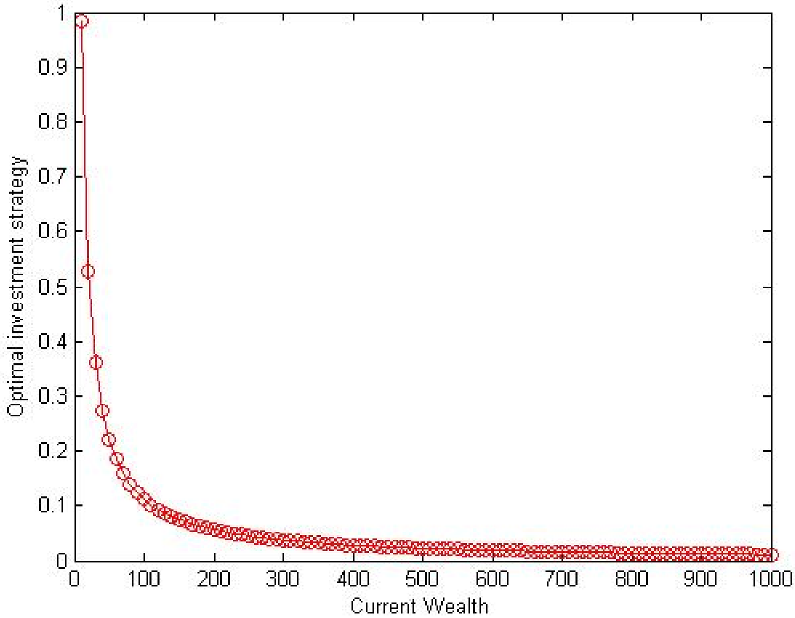

Figure 4.1: The optimal policy with respect to wealth for the power utility.

He/she looks somewhere he can invest his wealth with minimum or almost no risk at all to take on, while expecting for an absolute perfect return.

4.2.2 Simulation of value function

At this point we intend to study graphically how the value function behaves in relation to time and the wealth with the same market parameters used above.

The value function decreases with time and wealth. The observations show that, the value function does not decrease exactly to zero, yet it reaches a certain point where it shows some unnoticeable changes with respect to wealth, while continues to decrease exponentially with the increase in time. The surface described so far in the figure 4.2 shows a nonlinear relationship between the the value function and the time and wealth as well.

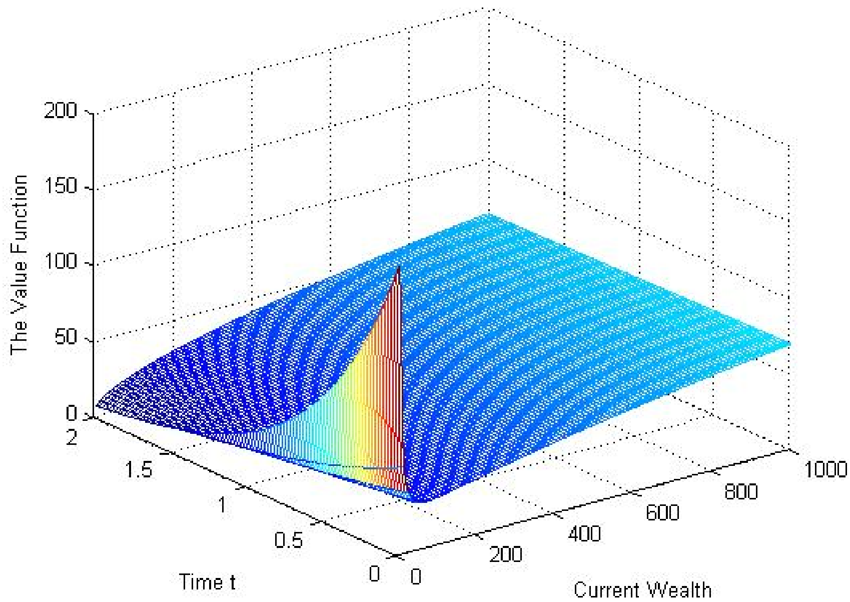

Figure 4.2: The value function with respect to time and wealth for the power utility.

4.3 Simulation in the case of exponential utility

Now we consider the results obtained when exponential utility used as the measure of the investor's behaviour in the market. We consider the market parameters $\mu = 10.54, \kappa = 0.3, \sigma = 0.97, r = 0.05, \Pi = 2,$ and $a = 1.$

4.3.1 Simulation of optimal strategy

We realize the graph of optimal investment strategy with respect to the wealth of the portfolio for the exponential utility.

The figure 4.3 shows that, the optimal strategy varies inversely with respect to wealth. As the wealth increases the optimal policy decreases. This result has the same implication as the one already discussed above for the power utility. Thats, the genuine investor reduces his proportions invested in risky assets and deposits them in bank accounts. This means that, the investor escapes from too much trading and now tries to find more time to get relaxed and avoid stresses.

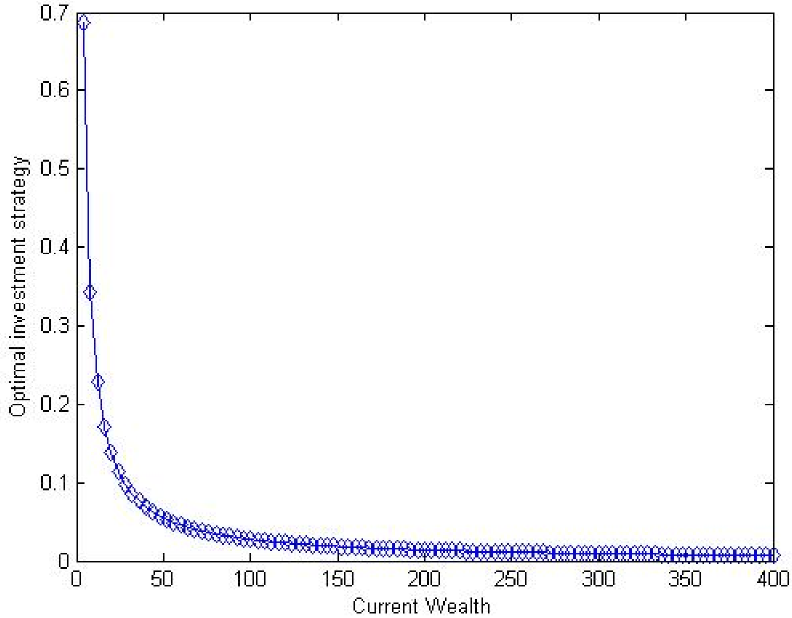

Figure 4.3: The optimal strategy with respect to current wealth for the exponential utility.

4.3.2 Simulation of value function

At this point we consider the realization of the value function with respect to time and wealth of the portfolio and the same market parameters used above for the exponential utility.

The figure 4.4 shows that the value function does not vary with respect to the wealth, but rather varies exponentially with respect to time. The value function increases negatively as the time advances with no effect from the wealth. The value function remains maximum no matter how wealth increases, however, that is not the case with time.

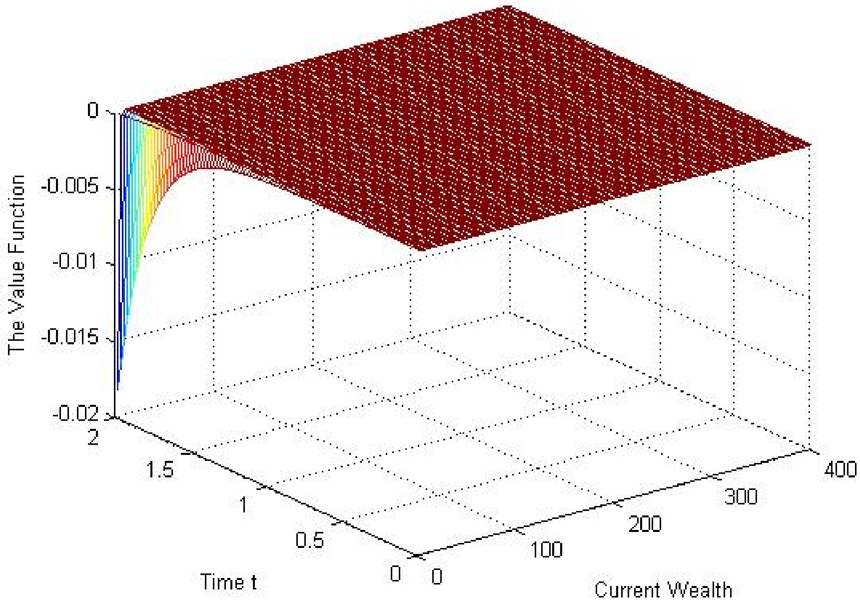

Figure 4.4: The value function with respect to time and wealth for the exponential utility.

CHAPTER FIVE
SUMMARY, CONCLUSION AND RECOMMENDATIONS

5.1 Introduction

In this chapter, we would like to give a summary of the entire work and a brief conclusion as well as to give some recommendations for possible extension of this study as the future research work.

5.2 Summary

In this work, we studied the investment processes, the management of the portfolio values and how an investor can achieve a portfolio with maximum returns for a given level of minimum risk. The objectives of the study were: To formulate a dynamic portfolio optimization problem for a suitable choice of utility functions as well as the Markov control policies. And to determine the optimal control strategy which maximizes the investors expected utility of wealth at some future time τ. In general, the work has been organized in five chapters as follows.

In chapter one, the introduction of the work has been given together with the goals of the study, statement and significance of the study as well as the hypothesis and methodologies in particular. The general presentation of the preliminary concepts, definitions and some theorems necessary and applicable for studying the problem were presented in parallel with the literature review in chapter two.

In chapter three the formulation of the problem of portfolio optimization were done and the task of determining the optimal investment strategy and the value function were dealt effectively with the application of specific utility functions.

We entirely assumed that the wealth controls are Markov controls and the portfolio strategy is self-financing and the corresponding wealth process is continuous and adapted process(meaning that, the change in the wealth is only due to changes in the bond or stock prices(i.e no external inflow or outflow of cash). The results obtained so far is such that, the optimal investment strategy found to be related to the wealth for both power and exponential utility functions.

The analysis of the results is done in chapter four. The simulation of the optimal investment strategy with respect to the wealth as well as the corresponding value function have been done for both power and exponential utility functions.The graphical study revealed that the optimal policy decreases with an increase in wealth. The implication is that, as the investor becomes richer, then he reduces his wealth proportions from the risky assets and deposits much in bank accounts. Lastly, chapter five includes summary and conclusion of the study and the recommendations for the future work.

5.3 Conclusion

In this dissertation, we have worked on portfolio management under the mean-reverting stock returns and the constant interest rate for bond returns. We have approached the problem of portfolio optimization by the theory of stochastic optimal control. We have managed to determine the optimal investment strategies and the value functions from the two theorems which have been stated and then proved for the power utility and exponential utility cases. The results however show that, the optimal investment rules are absolutely inversely related with the wealth and therefore rules out the popular investment allocation advice that, the more capital someone has the more he/she invests in risky assets for quick and better expected returns.

5.4 Recommendations

The investment problem studied so far involves only two assets, namely, bonds with the price at time t evolving exponentially with constant interest rate r and the stocks whose price at time t described by geometric mean-reversion model. The introduction of extra features such as consumption, human capital and transaction costs may bring model improvements and hence the optimal asset allocation choice.

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APPENDIX A
Codes for numerical simulation of optimal policy and value function

6.1 Codes for numerical simulation of optimal policy and value function for power utility case.

```
close all;
%Market Parameters;
k=0.3; %Mean reversion rate
mu=10.54; %Long-term mean equilibrium
sigma=0.97; %The degree of volatility responsible for randomness of the process
r=0.05; %The interest rate of the bond
alpha=0.5; %The parameter of power utility
%Optimal policy & Value function
n=100;
Pi=2;
theta=0:0.01:1;
x=linspace(10,1000,n);
t_1=2;
t=linspace(0,2,n);
[X T]=meshgrid(x,t);
b=alpha*(r+((k*(mu+1)-r)^2./((2*k*X/Pi)+sigma^2*(1-alpha)))); %beta of x
V=(X.^alpha/alpha).*exp(b.*(t_1-T)); %Value function
mesh(X,T,V)
Xlabel('Current Wealth');Ylabel('Time t');Zlabel('The Value Function');
figure
theta=(k*(mu+1)-r)./((2*k.*x/Pi)+sigma^2*(1-alpha)); %Optima policy
plot(x,theta,'ro-')
Ylabel('Optimal investment strategy');Xlabel('Current Wealth');
```

6.2 Codes for numerical simulation of optimal investment strategy and value function for exponential utility case.

```
close all;
%Market Parameters;
k=0.3; %Mean reversion rate
mu=10.54; %Long-term mean equilibrium
sigma=0.97; %The degree of volatility responsible for randomness of the process
r=0.05; %The interest rate of the bond
a=1; %The parameter of exponential utility
%Optimal policy & Value function
```

```
n=100;
Pi=2;
theta=0:0.01:1;
x=linspace(4,400,n);
t_1=2;
t=linspace(0,2,n);
[X T]=meshgrid(x,t);
g=-a*(r*X+1/2*((k*(mu+1)-r)^2./((2*k/Pi)+sigma^2*a))); %gamma of x
V=-exp(g.*(t_1-T)-a*X); %Value function
mesh(X,T,V)
Xlabel('Current Wealth');Ylabel('Time t');Zlabel('The Value Function');
figure
theta=(k*(mu+1)-r)./(((2*k/Pi)+sigma^2*a)*x); %Optima policy
plot(x,theta,'bd-')
Ylabel('Optimal investment strategy');Xlabel('Current Wealth');
```